Fermions in combinatorics: random permutations and partitions

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Determinantal point processes and fermions, Lille
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Outline

1. Introduction: the Longest Increasing Subsequence problem
2. Fermionic Fock space
3. From fermions to partitions: the discrete Bessel kernel
4. Asymptotics
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1 Introduction: the Longest Increasing Subsequence problem

2 Fermionic Fock space

3 From fermions to partitions: the discrete Bessel kernel

4 Asymptotics
Let us consider a uniform random permutation in $S_n$. What can be said about the length $L_n$ of a longest increasing subsequence?

Example: for $\sigma = (3, 1, 6, 7, 2, 5, 4)$, we have $L(\sigma) = 3$. 
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Some history of the LIS problem

- The problem was formulated by Ulam (1961) who suggested investigating it using Monte Carlo simulations and observed that $L_n$ should be of order $\sqrt{n}$.
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- It was then popularized by Hammersley (1972) who introduced a nice graphical method (closely related with the RSK correspondence) and proved that $L_n/\sqrt{n}$ converges in probability to a constant $c \in [\pi/2, e]$. 

Vershik-Kerov and Logan-Shepp (1977) proved independently that $c = 2$, as a consequence of a more general limit shape theorem for the Plancherel measure on partitions (to be defined).

Baik-Deift-Johansson (1999) proved the most precise result $P(L_n - 2\sqrt{n}/n^{1/6} \leq s) = F_{\text{GUE}}(s)$, $n \to \infty$ where $F_{\text{GUE}}$ is the Tracy-Widom GUE distribution.

See Romik’s book for a detailed account of this fascinating story, and Kammoun’s recent paper for extensions to other families of random permutations (universality).
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Plane partitions

Lozenge tiling

Plane partition

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Introduction: the Longest Increasing Subsequence problem

Fermionic Fock space

From fermions to partitions: the discrete Bessel kernel

Asymptotics
Fermions and Maya diagrams

Think of a collection of boxes labeled by the half-integers ("energy levels", positive or negative):

\[
\begin{array}{cccccccccccc}
\ \\
\ \\
\ \\
\ \\
\ \\
\ \\
\ \\
\ \\
\ \\
\ \\
\ \\
\end{array}
\]

\[
\begin{array}{cccccccccccc}
\ldots & -\frac{9}{2} & -\frac{7}{2} & -\frac{5}{2} & -\frac{3}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{3}{2} & \frac{5}{2} & \frac{7}{2} & \frac{9}{2} & \ldots
\end{array}
\]
Fermions and Maya diagrams

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\begin{array}{ccccccc}
\bullet & \bullet & \bullet & \circ & \bullet & \circ & \vdots \\
-9/2 & -7/2 & -5/2 & -3/2 & -1/2 & 1/2 & 3/2 \\
\end{array}
\]

Each box may contain at most one particle (\(\bullet\)). No particle = "hole" (\(\circ\)).
Fermions and Maya diagrams

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Any other Maya diagram is obtained by a finite number of operations:

- adding a particle with positive energy
- removing a particle with negative energy
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Any other Maya diagram is obtained by a finite number of operations:

- adding a particle with positive energy
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(total energy increases in both cases!)
Fock space

The fermionic Fock space $\mathcal{F}$ is the Hilbert space with basis index by Maya diagrams.

A element of $\mathcal{F}$ represents the wave function of a system of (infinitely) many fermions.

There is an underlying Hilbert space $\mathcal{H}_1$ describing the possible states of one particle, whose basis is indexed by half-integers.

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\[
\begin{align*}
| m \rangle &= | 7/2 \rangle \wedge | 1/2 \rangle \wedge | -3/2 \rangle \wedge | -7/2 \rangle \wedge | -9/2 \rangle \wedge \cdots \\
\end{align*}
\]

A Maya diagram may be thought as an infinite wedge product (Slater determinant) of one-particle basis states, which has a finite total energy.
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Physical relevance: description of the low-energy excitations of a gapless system of many fermions (1D). Energy 0 corresponds to the Fermi level.
Observables

Let $m$ be a Maya diagram and $|m\rangle$ the corresponding basis vector in $\mathcal{F}$. We consider observables that are diagonal in this basis.

Particle number operators: for $k \in \mathbb{Z}' = \mathbb{Z} + 1/2$, $N_k |m\rangle = \begin{cases} |m\rangle & \text{if } m \text{ has a particle at position } k, \\ 0 & \text{if } m \text{ has a hole at position } k \end{cases}$.

Charge/energy operators:

\[ C = \sum_{k \in \mathbb{Z}'} :N_k: \]
\[ H = \sum_{k \in \mathbb{Z}'} k :N_k: \]
where we set $:N_k: = N_k - \langle \emptyset | N_k | \emptyset \rangle$. Note that $H \geq 0$.

Next we define an important family of (nonhermitian) operators, the creation and annihilation operators $\psi_k$ and $\psi_k^*$ for $k \in \mathbb{Z}'$. 
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Creation and annihilation operators

\[ \psi_{9/2} |m\rangle = |m'\rangle \]

\( m \) \hspace{1cm} \( m' \)

\[ \psi_{9/2} \]

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Creation and annihilation operators

\[ \psi_{5/2} |m\rangle = - |m'\rangle \]
Creation and annihilation operators

\[ \psi_{7/2} |m\rangle = 0 \]
Creation and annihilation operators

\[ \psi^* \frac{7}{2} \mid m \rangle = \mid m' \rangle \]
Creation and annihilation operators

$$m$$

$$m'$$

$$\psi^*_{1/2} |m\rangle = - |m'\rangle$$
Creation and annihilation operators

\[ \psi^* \frac{1}{\sqrt{2}} |m\rangle = 0 \]
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We have the canonical anticommutation relations (CAR)

$$\{\psi_k, \psi_\ell^*\} = \delta_{k,\ell}, \quad \{\psi_k, \psi_\ell\} = \{\psi_\ell^*, \psi_k^*\} = 0$$

where $\{A, B\} := AB + BA$ denotes the anticommutator.
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We also have

$$N_k = \psi_k \psi_k^*, \quad \psi_k |\emptyset\rangle = \psi_k^* |\emptyset\rangle = 0 \text{ for } k < 0.$$

Any operator can be expressed in terms of the creation/annihilation operators!
Bilinear operators

Using the CAR, we see that the operators $\psi_i \psi_j^*$ form a Lie algebra:

$$[\psi_i \psi_j^*, \psi_k \psi_\ell^*] = \delta_{j,k} \psi_i \psi_\ell^* - \delta_{i,\ell} \psi_k \psi_j^*$$

isomorphic to $\mathfrak{gl}(\infty)$. Here $[A, B] := AB - BA$ is the ordinary commutator.
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Every operator acting on the one-particle Hilbert space $\mathcal{H}_1$ can be promoted as an operator on $F$ (second quantization). Of course there are more general operators on $F$ such as two-particle operators $N_k N_{k'} = \psi_k \psi_k^* \psi_{k'} \psi_{k'}^*$, etc.
Bosonic operators

Of particular interest are the **bosonic operators**

\[ \alpha_n := \sum_{k \in \mathbb{Z}'} \psi_k - n \psi_k^*, \quad n \in \mathbb{Z} \setminus \{0\} \]

whose action makes sense on \( F \) (finitely many terms contribute when acting on \( |m\rangle \)).
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\[ [\alpha_n, \alpha_m] = n \delta_{n,-m} \]

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and $\alpha_n = \alpha_{-n}^*$. (The boson-fermion correspondence states that the operators $\psi_k, \psi_k^*$ can be reconstructed from the bosonic operators.)
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In the following we will mostly consider the bosonic operators \( \alpha = \alpha_1 \) and \( \alpha^* = \alpha_{-1} \). The hermitian operator \( \alpha + \alpha^* \) describes a “hopping” dynamics.
Wick’s lemma

Before we move on to the connection with the LIS problem, let us make that with determinantal processes. Let $\langle O \rangle := \langle \emptyset | O | \emptyset \rangle$ denote the vacuum expectation value of an operator $O$. 

Let $\phi_1, \phi_3, \ldots, \phi_{2n-1}$ denote linear combinations of the $\psi_k$'s and $\phi^*_2, \phi^*_4, \ldots, \phi^*_{2n}$ denote linear combinations of the $\psi^*_k$'s. Then we have

$$
\langle \phi_1 \phi^*_2 \phi_3 \phi^*_4 \cdots \phi_{2n-1} \phi^*_{2n} \rangle = \det \left[ C_{i,j} \right]_{i,j=1}^{n}
$$

where

$$
C_{i,j} = \begin{cases} 
\langle \phi_{2i-1} \phi^*_j \rangle & \text{if } i \leq j \\
\langle \phi^*_j \phi_{2i-1} \rangle & \text{if } i > j
\end{cases}
$$

("time-ordered correlator").

In fact it also holds with other expectations values ("quasi-free states"): $\langle m | O | m \rangle$ for any Maya diagram $m$, $\langle \emptyset | e^{it\tilde{H}} O e^{-it\tilde{H}} | \emptyset \rangle$ for any bilinear ("free") Hamiltonian $\tilde{H}$, the grand canonical finite-temperature e.v. $\frac{1}{Z} \text{Tr} \left( O e^{-\beta (H - \mu C)} \right)$...
Wick’s lemma

Before we move on to the connection with the LIS problem, let us make that with determinantal processes. Let \( \langle O \rangle := \langle \emptyset | O | \emptyset \rangle \) denote the vacuum expectation value of an operator \( O \).

**Wick’s lemma (see Gaudin 1960 for a simple proof using CAR)**

Let \( \varphi_1, \varphi_3, \ldots, \varphi_{2n-1} \) denote linear combinations of the \( \psi_k \)'s and \( \varphi^*_2, \varphi^*_4, \ldots, \varphi^*_2n \) denote linear combinations of the \( \psi^*_k \)'s. Then we have

\[
\langle \varphi_1 \varphi^*_2 \varphi_3 \varphi^*_4 \cdots \varphi_{2n-1} \varphi^*_2n \rangle = \det_{1 \leq i, j \leq n} C_{i,j}
\]

where \( C_{i,j} = \begin{cases} 
\langle \varphi_{2i-1} \varphi^*_{2j} \rangle & \text{if } i \leq j \\
-\langle \varphi^*_{2j} \varphi_{2i-1} \rangle & \text{if } i > j
\end{cases} \) ("time-ordered correlator").
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In fact it also holds with other expectations values (“quasi-free states”):

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- the grand canonical finite-temperature e.v. \( \frac{1}{Z} \text{Tr}(O e^{-\beta(H-\mu C)}) \)...
Wick’s lemma

Recall that $N_x = \psi_x \psi_x^*$ measures the particle number at position $x$. Then, for distinct positions $x_1, \ldots, x_n$ we get that

$$\langle N_{x_1} \cdots N_{x_n} \rangle = \det_{1 \leq i, j \leq n} K(x_i, x_j)$$

where $K(x, x') = \langle \psi_x \psi_{x'}^* \rangle$ is the correlation kernel.
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There is also a natural way to construct a time-extended process using time-dependent operators (Heisenberg picture):

$$N_x(t) = e^{it\tilde{H}} N_x e^{-it\tilde{H}}$$

with $\tilde{H}$ a free Hamiltonian. The key fact is that $N_x(t)$ remains a bilinear combination of creation/annihilation operators.
1. Introduction: the Longest Increasing Subsequence problem

2. Fermionic Fock space

3. From fermions to partitions: the discrete Bessel kernel

4. Asymptotics
There is a combinatorial correspondence between Maya diagrams and Young diagrams (aka integer partitions).
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\[ \alpha^* |\lambda\rangle = \sum |\lambda'\rangle \]

where the sum runs over all \( \lambda' \) obtained by adding a box to \( \lambda \).
Standard Young tableaux

Let $\lambda$ be a Young diagram with $n$ boxes.
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A standard Young tableau (SYT) of shape $\lambda$ is a numbering of the boxes of $\lambda$ by the integers $\{1, \ldots, n\}$ which is "increasing".
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Therefore

$$d_\lambda := \langle \lambda | (\alpha^*)^n | \emptyset \rangle$$

is equal to the number of SYT of shape $\lambda$. 
Connection with the LIS problem

It is known that permutations are closely related with Young diagrams/tableaux: the Robinson-Schensted correspondence states that there is a bijection between:

- permutations $\sigma$ of $\{1, \ldots, n\}$,
- and triples $(\lambda, T, T')$ where $\lambda$ is a Young diagram with $n$ boxes and $T, T'$ are two SYT of shape $\lambda$.

In this correspondence the length of a longest increasing subsequence $L(\sigma)$ is equal to the length $\lambda_1$ of the first row of $\lambda$. 

Therefore, the LIS problem becomes a question about the Plancherel measure on Young diagrams:

$$\text{Prob}(\lambda) = \frac{d_{2\lambda}}{n!}.$$ 

In turn, it becomes a question about Maya diagrams: $\lambda_1 < \ell$ iff the Maya diagram of $\lambda$ contains no particle in the interval $[\ell + 1/2, \infty)$. 

Jérémie Bouttier (CEA/ENS de Lyon)
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Poissonized Plancherel measure

It proves convenient to take the size $n$ to be a Poisson random variable, and consider the **poissonized Plancherel measure**

$$\text{Prob}(\lambda) = \frac{d^2}{|\lambda|!} x^{2|\lambda|} e^{-x^2}.$$ 

For $x \to \infty$ the size $|\lambda|$ concentrates around $x$. 
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$$\text{Prob}(\lambda) = \frac{d^2 \lambda}{|\lambda|!} x^2 |\lambda| e^{-x^2}.$$  

For $x \to \infty$ the size $|\lambda|$ concentrates around $x$. But we have

$$\frac{d\lambda}{|\lambda|!} x^{|\lambda|} = \langle \lambda | e^{x \alpha^*} | \emptyset \rangle$$

We recognize a quantum measurement with respect to the "coherent" state $e^{x \alpha^* - x^2 / 2} | \emptyset \rangle$. Wick's theorem holds hence we find that the associated (random) Maya diagram is a determinantal point process.
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We recognize a quantum measurement with respect to the “coherent” state $e^{x\alpha^*} - x^2/2 |\emptyset\rangle$. Wick’s theorem holds hence we find that the associated (random) Maya diagram is a determinantal point process.
The discrete Bessel kernel

Using the CAR it is possible to compute explicitly the correlation kernel:

\[ K(i, j) = \langle \emptyset | e^{x\alpha} \psi_i \psi_j^* e^{x\alpha^*} | \emptyset \rangle e^{-x^2} \]
\[ = \sum_{\ell < 0} J_{i-\ell}(2x) J_{j-\ell}(2x) \]

with \( J_n \) the Bessel function of the first kind.
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Using the CAR it is possible to compute explicitly the correlation kernel:

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K(i, j) = \langle \emptyset | e^{x\alpha_\psi} \psi_i \psi_j^* e^{x\alpha_\psi^*} | \emptyset \rangle e^{-x^2}
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with \( J_n \) the Bessel function of the first kind.

Here \( \nu_i := J_{i-\ell}(2x) \) may be thought as a one-particle eigenfunction:

\[
i \nu_i - x(\nu_{i-1} + \nu_{i+1}) = \ell \nu_i.
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with \( J_n \) the Bessel function of the first kind.

Here \( v_i := J_{i-\ell}(2x) \) may be thought as a one-particle eigenfunction:

\[ iv_i - x(v_{i-1} + v_{i+1}) = \ell v_i. \]

Thus \( K \) may be understood as the projector on the space of states with negative eigenvalue.
Outline

1 Introduction: the Longest Increasing Subsequence problem

2 Fermionic Fock space

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4 Asymptotics
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The following asymptotic analysis of the discrete Bessel kernel was done by Borodin, Okounkov and Olshanski (2000).
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First it is natural to analyze the one-point function $K(i, i)$. We let the poissonization parameter $x \to \infty$ keeping $y = i/x$ fixed:

$$\lim_{x \to \infty} K(xy, xy) = \rho(y) = \begin{cases} \arccos(y/2) & \text{if } y \in (-2, 2), \\ \frac{\arccos(y/2)}{\pi} & \text{if } y \leq -2, \\ 0 & \text{if } y \geq 2. \end{cases}$$
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We recover the Vershik-Kerov-Logan-Shepp limit shape.
Bulk asymptotics: discrete sine kernel

More generally we have

$$\lim_{x \to \infty} K(i, j) = \frac{\sin(\rho(y) \pi d)}{\pi d}.$$  

It tends to 0 as $d \to \infty$: decorrelation.
Bulk asymptotics: discrete sine kernel

More generally we have

$$\lim_{x \to \infty} K(i, j) = \frac{\sin(\rho(y) \pi d)}{\pi d}.$$

It tends to 0 as $d \to \infty$: decorrelation.

This bulk limit is universal in discrete combinatorial models (dimers...).
Edge asymptotics

Now let us zoom on the edge of the limit shape, where $\rho$ vanishes. Here the typical distance between particles is of order $x^{-1/3}$ so we need to rescale:

$$\lim_{x \to \infty} x^{1/3} K(2x + sx^{1/3}, 2x + tx^{1/3}) = K_{Ai}(s, t)$$

where $K_{Ai}$ is the Airy kernel

$$K_{Ai}(s, t) = \int_0^\infty \text{Ai}(s + u) \text{Ai}(t + u) du$$

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This essentially proves the BDJ theorem:

\[
\mathbb{P} \left( \lambda_1 \leq 2x + sx^{1/3} \right) = \det(I - K)_{\ell^2([2x+sx^{1/3}], \infty)} \\
\rightarrow \det(I - K_{Ai})_{L^2(s, \infty)} = F_{GUE}(s).
\]

(The first equality is a general property of DPPs, the convergence of Fredholm determinants is easy to justify, and the last equality is known.)
Conclusion

We have seen how to prove the Baik-Deift-Johansson theorem using fermions. This approach is essentially due to Okounkov and collaborators in the 2000’s.

My own contributions, not discussed in this talk, in the more general context of Schur processes:

- the case of positive temperature (involving the finite-temperature Airy kernel), see arXiv:1807.09022 [math-ph],
- the “free boundary case” (involving pfaffian point processes, the Tracy-Widom GOE/GSE distributions, and “superconducting” fermionic states), see arXiv:1704.05809 [math.PR].