Fermions in combinatorics: random permutations and partitions

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Determinantal point processes and fermions, Lille
7 February 2019

Outline

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The Longest Increasing Subsequence problem

Let us consider a uniform random permutation in S_n. What can be said about the length L_n of a longest increasing subsequence?

Example: for $\sigma = (3, 1, 6, 7, 2, 5, 4)$, we have $L(\sigma) = 3$.

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- Baik-Deift-Johansson (1999) proved the most precise result

$$\mathbb{P}\left(\frac{L_n - 2\sqrt{n}}{n^{1/6}} \leq s\right) = F_{GUE}(s), \qquad n \to \infty$$

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- See Romik's book for a detailed account of this fascinating story, and Kammoun's recent paper for extensions to other families of random permutations (universality).

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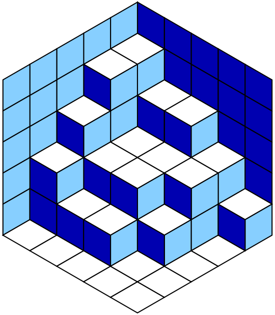
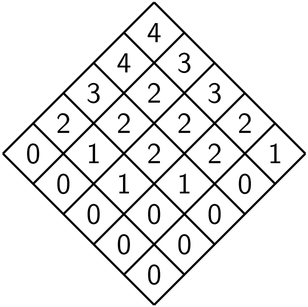
Plane partitions

Lozenge tiling



Plane partition



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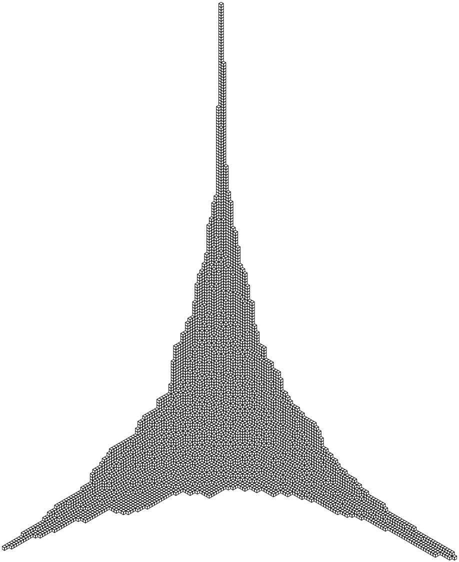

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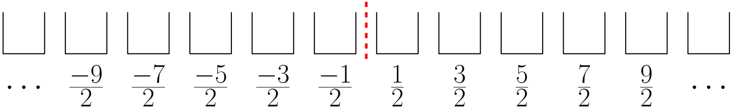
Fermions and Maya diagrams

Think of a collection of boxes labeled by the half-integers (“energy levels”, positive or negative):



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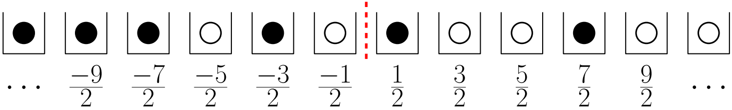

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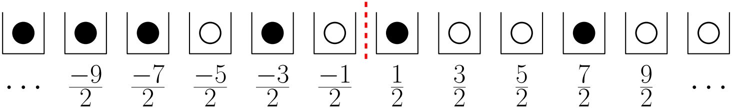

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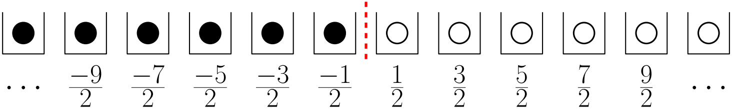

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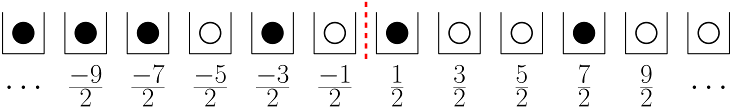

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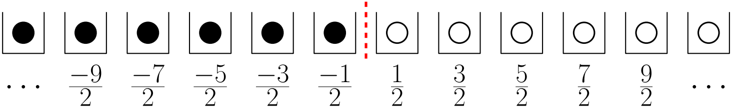

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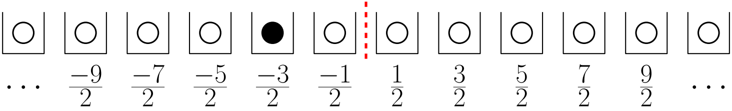
(total energy increases in both cases!)

Fock space

The fermionic Fock space $\mathcal{F}$ is the Hilbert space with basis index by Maya diagrams.

A element of $\mathcal{F}$ represents the wave function of a system of (infinitely) many fermions.

There is an underlying Hilbert space $\mathcal{H}_1$ describing the possible states of one particle, whose basis is indexed by half-integers.



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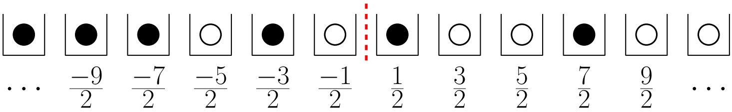

$$|m\rangle = |7/2\rangle \wedge |1/2\rangle \wedge |-3/2\rangle \wedge |-7/2\rangle \wedge |-9/2\rangle \wedge \cdots$$

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Physical relevance: description of the low-energy excitations of a gapless system of many fermions (1D). Energy 0 corresponds to the Fermi level.

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Let m be a Maya diagram and $|m\rangle$ the corresponding basis vector in $\mathcal{F}$. We consider observables that are diagonal in this basis.

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$$N_k|m\rangle = \begin{cases} |m\rangle & \text{if } m \text{ has a particle at position } k, \\ 0 & \text{if } m \text{ has a hole at position } k. \end{cases}$$

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- Charge/energy operators:

$$C = \sum_{k \in \mathbb{Z}'} {:}N_k{:}, \qquad H = \sum_{k \in \mathbb{Z}'} k \, {:}N_k{:}$$

where we set ${:}N_k{:} = N_k - \langle \emptyset | N_k | \emptyset \rangle$. Note that $H \geq 0$!

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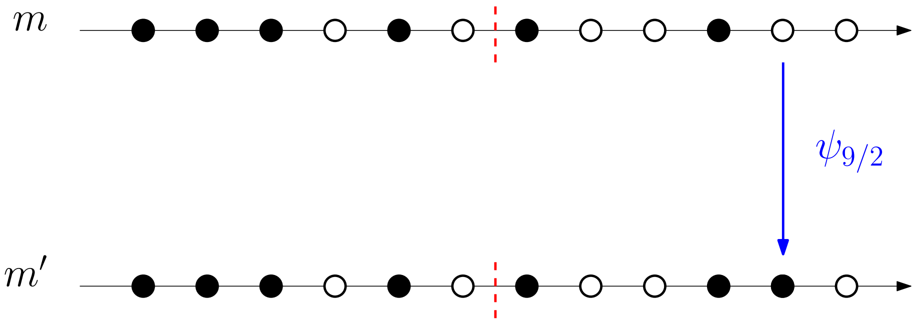
Next we define an important family of (nonhermitian) operators, the creation and annihilation operators ψ_k and ψ_k^* for $k \in \mathbb{Z}'$.

Creation and annihilation operators



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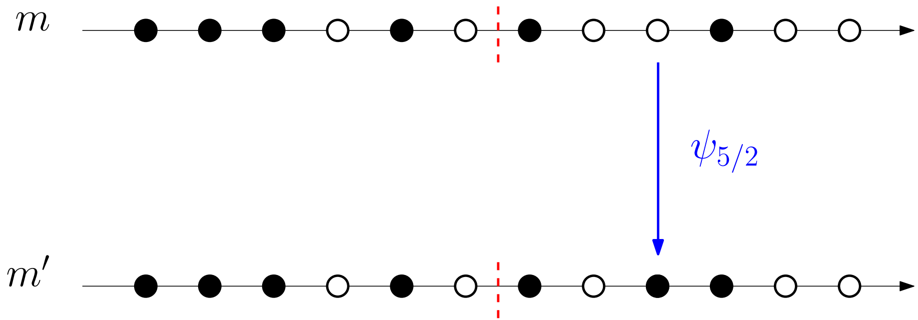

$$\psi_{5/2}|m\rangle = -|m'\rangle$$

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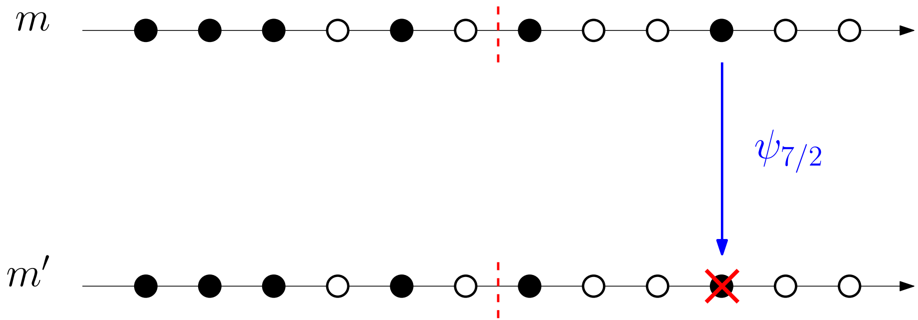

$$\psi_{7/2}|m\rangle = 0$$

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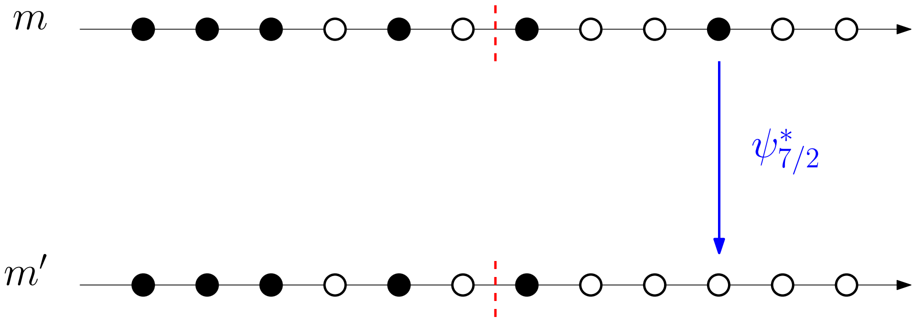

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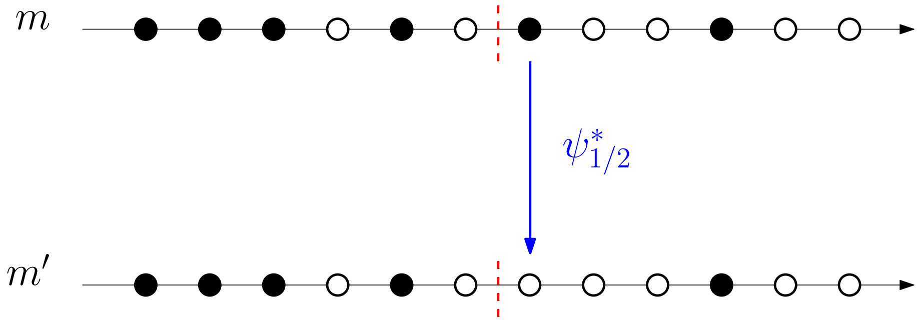

$$\psi^*_{1/2}|m\rangle = -|m'\rangle$$

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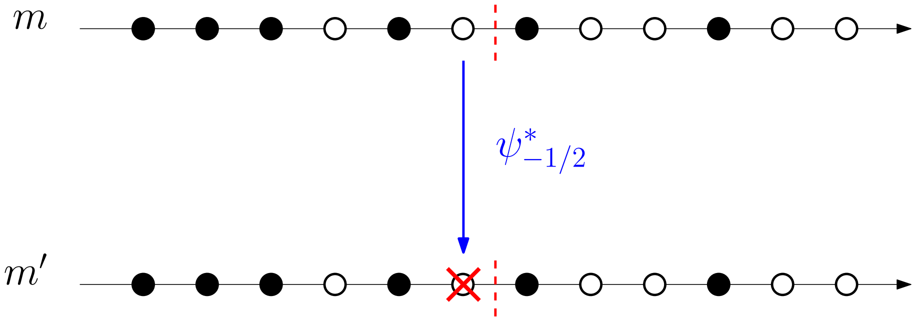

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We have the canonical anticommutation relations (CAR)

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We also have

$$N_k = \psi_k \psi_k^*, \qquad \psi_k|\emptyset\rangle = \psi_{-k}^*|\emptyset\rangle = 0 \text{ for } k < 0.$$

Any operator can be expressed in terms of the creation/annihilation operators!

Bilinear operators

Using the CAR, we see that the operators $\psi_i \psi_j^*$ form a Lie algebra:

$$[\psi_i \psi_j^*, \psi_k \psi_\ell^*] = \delta_{j,k} \psi_i \psi_\ell^* - \delta_{i,\ell} \psi_k \psi_j^*$$

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Bosonic operators

Of particular interest are the bosonic operators

$$\alpha_n := \sum_{k \in \mathbb{Z}'} \psi_{k-n} \psi_k^*, \qquad n \in \mathbb{Z} \setminus \{0\}$$

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(The boson-fermion correspondence states that the operators ψ_k, ψ_k^* can be reconstructed from the bosonic operators.)

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Of particular interest are the bosonic operators

$$\alpha_n := \sum_{k\in\mathbb{Z}'} \psi_{k-n}\psi_k^*, \qquad n \in \mathbb{Z}\setminus\{0\}$$

whose action makes sense on $\mathcal{F}$ (finitely many terms contribute when acting on $|m\rangle$). For $n=0$, $\alpha_0 = \sum_{k\in\mathbb{Z}'} {:}\psi_k\psi_k^*{:}$ is the charge operator C. They satisfy the canonical commutation relations

$$[\alpha_n, \alpha_m] = n\delta_{n,-m}$$

and $\alpha_n = \alpha_{-n}^*$.
(The boson-fermion correspondence states that the operators ψ_k, ψ_k^* can be reconstructed from the bosonic operators.)
In the following we will mostly consider the bosonic operators $\alpha = \alpha_1$ and $\alpha^* = \alpha_{-1}$. The hermitian operator $\alpha + \alpha^*$ describes a "hopping" dynamics.

Wick's lemma

Before we move on to the connection with the LIS problem, let us make that with determinantal processes. Let $\langle \mathcal{O} \rangle := \langle \emptyset | \mathcal{O} | \emptyset \rangle$ denote the vacuum expectation value of an operator $\mathcal{O}$.

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Wick's lemma (see Gaudin 1960 for a simple proof using CAR)

Let $\varphi_1, \varphi_3, \ldots, \varphi_{2n-1}$ denote linear combinations of the ψ_k's and $\varphi_2^*, \varphi_4^*, \ldots, \varphi_{2n}^*$ denote linear combinations of the ψ_k^*'s. Then we have

$$\langle \varphi_1 \varphi_2^* \varphi_3 \varphi_4^* \cdots \varphi_{2n-1} \varphi_{2n}^* \rangle = \det_{1 \le i,j \le n} C_{i,j}$$

where $C_{i,j} = \begin{cases} \langle \varphi_{2i-1} \varphi_{2j}^* \rangle & \text{if } i \le j \\ -\langle \varphi_{2j}^* \varphi_{2i-1} \rangle & \text{if } i > j \end{cases}$ ("time-ordered correlator").

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In fact it also holds with other expectations values ("quasi-free states"):

- $\langle m | \mathcal{O} | m \rangle$ for any Maya diagram m,
- $\langle \emptyset | e^{it\tilde{H}} \mathcal{O} e^{-it\tilde{H}} | \emptyset \rangle$ for any bilinear ("free") Hamiltonian $\tilde{H}$,
- the grand canonical finite-temperature e.v. $\frac{1}{Z} \mathrm{Tr}(\mathcal{O} e^{-\beta(H - \mu C)})$...

Wick's lemma

Recall that $N_x = \psi_x \psi_x^*$ measures the particle number at position x. Then, for distinct positions $x_1, \ldots, x_n$ we get that

$$\langle N_{x_1} \cdots N_{x_n} \rangle = \det_{1 \leq i,j \leq n} K(x_i, x_j)$$

where $K(x, x') = \langle \psi_x \psi_{x'}^* \rangle$ is the correlation kernel.

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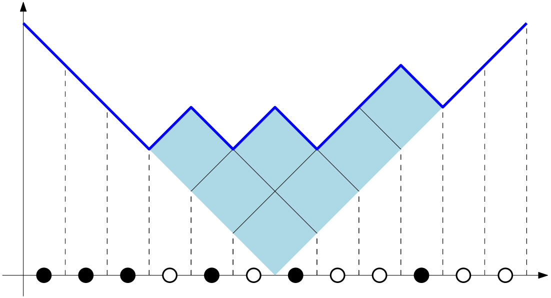
There is also a natural way to construct a time-extended process using time-dependent operators (Heisenberg picture):

$$N_x(t) = e^{it\tilde{H}} N_x e^{-it\tilde{H}}$$

with $\tilde{H}$ a free Hamiltonian. The key fact is that $N_x(t)$ remains a bilinear combination of creation/annihilation operators.

Outline

From Maya diagrams to Young diagrams



There is a combinatorial correspondence between Maya diagrams and Young diagrams (aka integer partitions).

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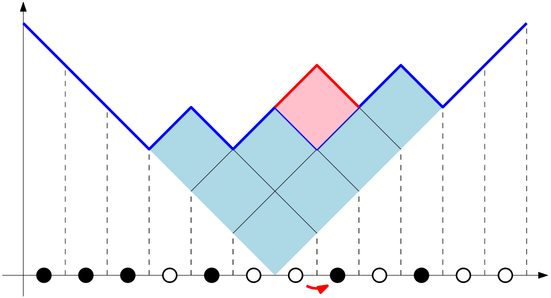

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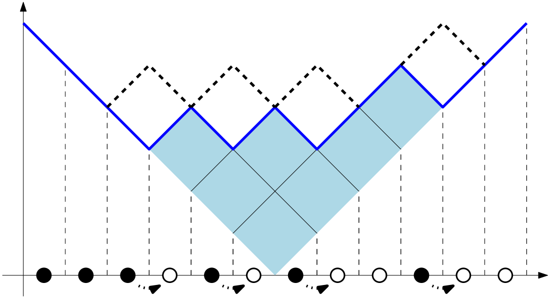


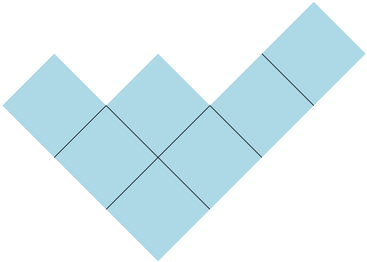
There is a combinatorial correspondence between Maya diagrams and Young diagrams (aka integer partitions).
In this picture, a particle hopping one site to the right corresponds to a box being added to the Young diagram.
Therefore we have

$$\alpha^*|\lambda\rangle = \sum |\lambda'\rangle$$

where the sum runs over all λ' obtained by adding a box to λ.

Standard Young tableaux

Let λ be a Young diagram with n boxes.



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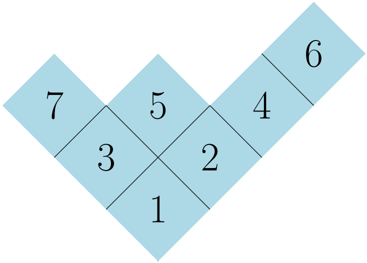

A standard Young tableau (SYT) of shape λ is a numbering of the boxes of λ by the integers $\{1, \ldots, n\}$ which is "increasing".

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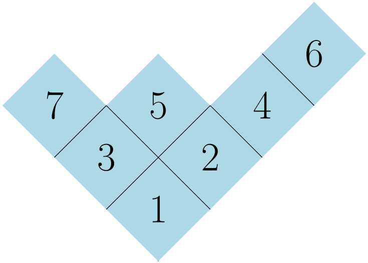

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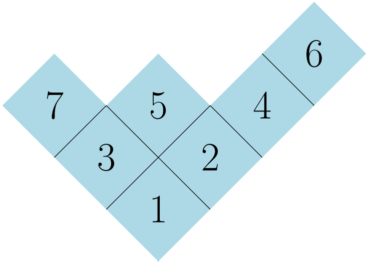

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Therefore

$$d_\lambda := \langle \lambda | (\alpha^*)^n | \emptyset \rangle$$

is equal to the number of SYT of shape λ.

Connection with the LIS problem

It is known that permutations are closely related with Young diagrams/tableaux: the Robinson-Schensted correspondence states that there is a bijection between:

- permutations σ of $\{1, \ldots, n\}$,
- and triples (λ, T, T') where λ is a Young diagram with n boxes and T, T' are two SYT of shape λ.

In this correspondence the length of a longest increasing subsequence $L(\sigma)$ is equal to the length λ_1 of the first row of λ.

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In turn, it becomes a question about Maya diagrams: $\lambda_1 < \ell$ iff the Maya diagram of λ contains no particle in the interval $[\ell + 1/2, \infty)$.

Poissonized Plancherel measure

It proves convenient to take the size n to be a Poisson random variable, and consider the poissonized Plancherel measure

$$\mathrm{Prob}(\lambda) = \frac{d_\lambda^2}{|\lambda|!} x^{2|\lambda|} e^{-x^2}.$$

For $x \to \infty$ the size $|\lambda|$ concentrates around x.

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We recognize a quantum measurement with respect to the "coherent" state $e^{x\alpha^* - x^2/2} | \emptyset \rangle$. Wick's theorem holds hence we find that the associated (random) Maya diagram is a determinantal point process.

The discrete Bessel kernel

Using the CAR it is possible to compute explicitly the correlation kernel:

$$\begin{aligned} K(i,j) &= \langle \emptyset | e^{x\alpha} \psi_i \psi_j^* e^{x\alpha^*} | \emptyset \rangle e^{-x^2} \\ &= \sum_{\ell<0} J_{i-\ell}(2x) J_{j-\ell}(2x) \end{aligned}$$

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Thus K may be understood as the projector on the space of states with negative eigenvalue.

Outline

Asymptotics

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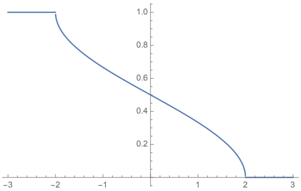
First it is natural to analyze the one-point function $K(i,i)$. We let the poissonization parameter $x \to \infty$ keeping $y = i/x$ fixed:

$$\lim_{x\to\infty} K(xy,xy) = \rho(y) = \begin{cases} \frac{\arccos(y/2)}{\pi} & \text{if } y \in (-2,2), \\ 1 & \text{if } y \leq -2, \\ 0 & \text{if } y \geq 2. \end{cases}$$



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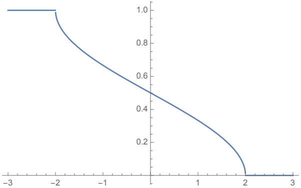
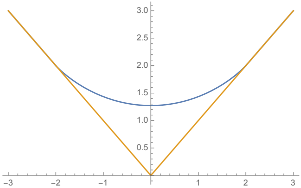
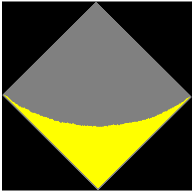






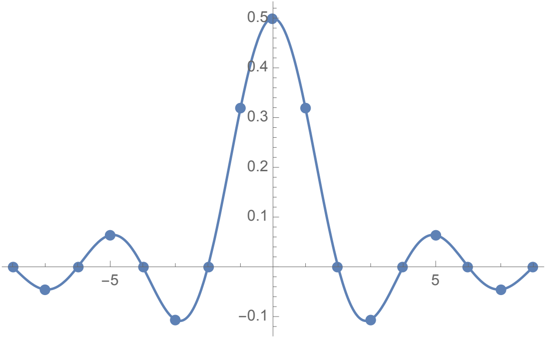
We recover the Vershik-Kerov-Logan-Shepp limit shape.

Bulk asymptotics: discrete sine kernel

More generally we have

$$\lim_{\substack{x\to\infty \\ i/x,\, j/x \to y \\ j-i=d \text{ fixed}}} K(i,j) = \frac{\sin(\rho(y)\pi d)}{\pi d}.$$

It tends to 0 as $d \to \infty$: decorrelation.



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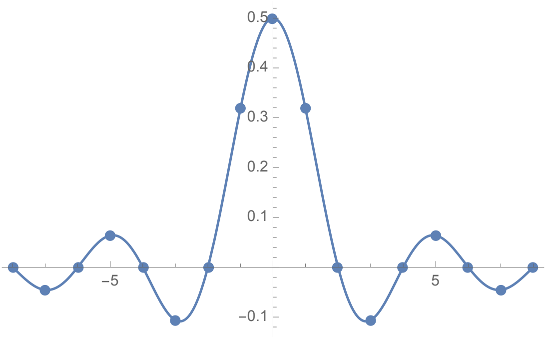

This bulk limit is universal in discrete combinatorial models (dimers...).

Edge asymptotics

Now let us zoom on the edge of the limit shape, where ρ vanishes. Here the typical distance between particles is of order $x^{-1/3}$ so we need to rescale:

$$\lim_{x\to\infty} x^{1/3} K(2x + sx^{1/3}, 2x + tx^{1/3}) = K_{\mathrm{Ai}}(s,t)$$

where K_{Ai} is the Airy kernel

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This essentially proves the BDJ theorem:

$$\begin{aligned}\mathbb{P}\left(\lambda_1 \leq 2x + sx^{1/3}\right) &= \det(I-K)_{\ell^2(\lfloor 2x+sx^{1/3}\rfloor,\infty)} \\ &\to \det(I-K_{\mathrm{Ai}})_{L^2(s,\infty)} = F_{GUE}(s).\end{aligned}$$

(The first equality is a general property of DPPs, the convergence of Fredholm determinants is easy to justify, and the last equality is known.)

Conclusion

We have seen how to prove the Baik-Deift-Johansson theorem using fermions. This approach is essentially due to Okounkov and collaborators in the 2000's.

My own contributions, not discussed in this talk, in the more general context of Schur processes:

- the case of positive temperature (involving the finite-temperature Airy kernel), see arXiv:1807.09022 [math-ph],
- the "free boundary case" (involving pfaffian point processes, the Tracy-Widom GOE/GSE distributions, and "superconducting" fermionic states), see arXiv:1704.05809 [math.PR].