The art of being a fermion in a sea of many possibilities

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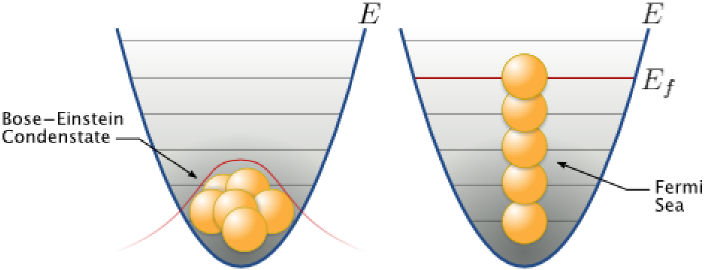
DPP-Fermions, Lille

Outline

1. Quantum statistics — why fermions?
2. Density functionals (Thomas-Fermi Approximation)
3. 2D magnetic, Laughlin and clustering states

Quantum statistics (in 3D)



Cold **bosons** Cold **fermions**

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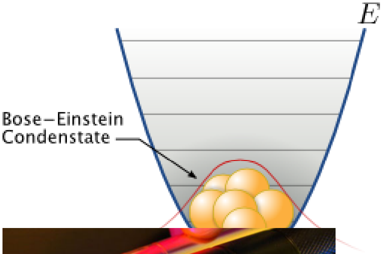
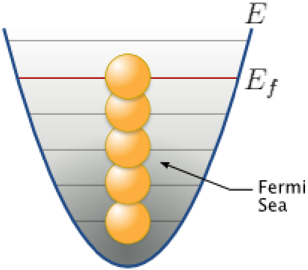


force carriers (fluffy)



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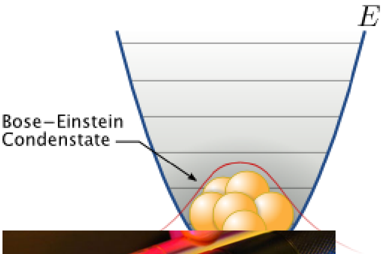

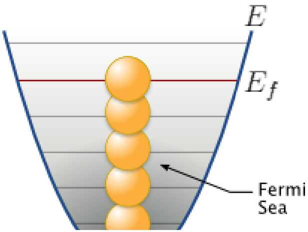

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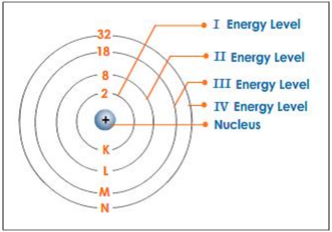

matter (stable)

Quantum statistics: Why fermions?

A **quantum wave function** $\Psi\colon (\mathbb{R}^3)^N \to \mathbb{C}$ subject to **symmetry**

$$\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_k, \ldots, \mathbf{x}_N) = \pm\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_k, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_N)$$

Observable: $|\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_k, \ldots, \mathbf{x}_N)|^2$ prob. distribution

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States of being + Identity

(not the only result)

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Generally a result of:

States of being + Identity

#pluralism + #oneness ⇒ #diversity

(not the only result)

Let us take the perspective of a particle...

Quantum statistics: configurations

Different **states of being** – configurations – potentiality

Configuration space $\mathscr{C}_1$ for a single particle

Ex1: $\mathscr{C}_1 = \{-1, 1\}$
Ex2: $\mathscr{C}_1 = \mathbb{R}^d$

Connectivity important!
"How may I shift from one state of being to another?"

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What is a particle?

- A representation/manifestation of some symmetry?
- A probability distribution on $\mathscr{C}_1$?
- An observable state subject to certain operations (operators)?
- A player on a game board defined by $\mathscr{C}_1$? (some dynamics)

$$L^2(\mathscr{C}_1), \quad \hat{X} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \hat{H} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad U(t) = e^{it\hat{H}}$$

Quantum statistics: identity / parallellism

Identity crisis (or just loneliness?) — **“What if I were many?”**

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$\mathscr{C}_1^{\times N}$		$\mathscr{C}_1^{\times N}/\sim$		$(\mathscr{C}_1^{\times N} \setminus \text{coinc.})/\sim$
distinguishable	vs	**identical**	vs	**identical but distinct**

$$(\mathbf{x}_1, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_k, \ldots, \mathbf{x}_N) \sim (\mathbf{x}_1, \ldots, \mathbf{x}_k, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_N)$$

"Quantum logic": if there is **no** way (no observable) to distinguish states/configurations – then they must be identified!

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$$\{\mathbf{x}_1, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_k, \ldots, \mathbf{x}_N\} \subseteq \mathscr{C}_1$$

"Quantum logic": if there is **no** way (no observable) to distinguish states/configurations – then they must be identified!

$\Rightarrow$ **Configuration space for N identical particles:**

$$\mathscr{C}_N := (\mathscr{C}_1^{\times N} \setminus \text{coinc.})/\sim \;\cong\; \{N\text{-point subsets of } \mathscr{C}_1\}$$

[Gibbs 1870's; Leinaas, Myrheim, 1977]

Quantum statistics: two identical particles in $\mathbb{R}^d$

$$\mathscr{C}_2 \;=\; \mathbb{R}^d \times \mathbb{R}^d/\!\sim \;\cong\; \mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{S}^{d-1}/\!\sim$$

Identification $(\mathbf{x}_1, \mathbf{x}_2) \sim (\mathbf{x}_2, \mathbf{x}_1)$
Center-of-mass coordinate $\mathbf{X} := \frac{1}{2}(\mathbf{x}_1 + \mathbf{x}_2)$
Relative coordinate $\mathbf{r} := \mathbf{x}_1 - \mathbf{x}_2 = r\mathbf{n}$, $|\mathbf{n}| = 1$, $\mathbf{n} \sim -\mathbf{n}$

Connectivity: Inherited from a choice of dynamics, say

$$H_2 = \frac{1}{2m}\left(\mathbf{p}_1^2 + \mathbf{p}_2^2\right) = \frac{1}{4m}\mathbf{p}_{\mathbf{X}}^2 + \frac{1}{m}\mathbf{p}_{\mathbf{r}}^2 = \frac{1}{4m}\mathbf{p}_{\mathbf{X}}^2 + \frac{1}{m}\left(p_r^2 + \frac{1}{r^2}\mathbf{p}_{\mathbf{n}}^2\right)$$

Consider group of **continuous loops** in $\mathscr{C}_2$ (modulo simple loops)

$$[0,1] \ni t \mapsto \mathbf{n}(t) \in \mathbb{S}^{d-1}, \quad \mathbf{n}(1) = \pm\mathbf{n}(0)$$

$$\pi_1(\mathscr{C}_2) \cong \pi_1(\mathbb{S}^{d-1}/\sim) \cong \begin{cases} 1, & d = 1, \\ \mathbb{Z}, & d = 2, \\ \mathbb{Z}_2, & d \geq 3. \end{cases}$$

Quantum statistics: N identical particles in $\mathbb{R}^d$

Configuration space $\mathscr{C}_N = \{N\text{-point subsets of } \mathbb{R}^d\}$.
Typical dynamics

$$H_N = \sum_{j=1}^{N} \Big(\frac{1}{2m} \mathbf{p}_j^2 + V(\mathbf{x}_j) \Big) + \sum_{j<k} W(\mathbf{x}_j - \mathbf{x}_k)$$

Exchanges of particles are **continuous loops** in $\mathscr{C}_N$:

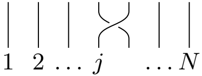
$$\{\text{loops in } \mathscr{C}_N \text{ up to homotopy}\} = \pi_1(\mathscr{C}_N) = \begin{cases} 1, & d = 1, \\ B_N, & d = 2, \\ S_N, & d \geq 3. \end{cases}$$

Quantum statistics: braid group

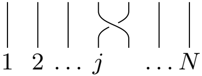
B_N is the **braid group** on N strands:

$$B_N = \left\langle \sigma_1, \ldots, \sigma_{N-1} : \sigma_j \sigma_{j+1} \sigma_j = \sigma_{j+1} \sigma_j \sigma_{j+1},\ \sigma_j \sigma_k = \sigma_k \sigma_j \right\rangle_{|j-k|>1}$$

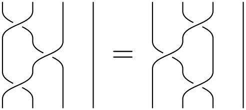
σ_j :



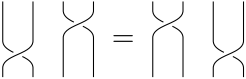
σ_j^{-1} :



Ex in B_4:



$$\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2$$



$$\sigma_1 \sigma_3 = \sigma_3 \sigma_1$$

If we add the relations $\sigma_j^2 = 1$ we obtain the **permutation group** S_N

Quantum statistics: exchange statistics

We may insist that a **local** definition of dynamics

$$\Omega \subseteq \mathscr{C}_N \quad \text{top. trivial} \Rightarrow \text{distinguishable} \Rightarrow$$

$$\hat{H}_N \text{ acting in some local Hilbert space } L^2(\Omega; \mathfrak{h})$$

extends to a **global** definition on $\mathscr{C}_N$ (a vector bundle with fiber $\mathfrak{h}$).

Additional information required upon gluing such local information: a **representation** of exchanges as operators

$$\rho \colon \pi_1(\mathscr{C}_N) \to \mathrm{U}(\mathfrak{h}).$$

Simplest case $\mathfrak{h} = \mathbb{C}$: $\rho(\sigma_j) = e^{i\theta_j} \in \mathrm{U}(1)$,

$$e^{i\theta_j} e^{i\theta_{j+1}} e^{i\theta_j} = e^{i\theta_{j+1}} e^{i\theta_j} e^{i\theta_{j+1}}$$

Exchange phase: $\theta_j = 0 \Rightarrow \rho = 1 \Rightarrow$ **bosons**,
$\theta_j = \pi \Rightarrow \rho = \mathrm{sign} \Rightarrow$ **fermions**, otherwise "**anyons**" (only in 2D)

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Quantum statistics: exclusion statistics

For bosons and fermions we can again extend to $\mathscr{C}^{\times N} \sim S_N \times \mathscr{C}_N$:

$$\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_k, \ldots, \mathbf{x}_N) = \pm\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_k, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_N)$$

$$\bigotimes_{\text{sym}}^{N} L^2(\mathscr{C}_1) \qquad \text{vs} \qquad \bigwedge^{N} L^2(\mathscr{C}_1)$$

Slater determinant:

$$(\psi_1 \wedge \ldots \wedge \psi_N)(\mathbf{x}_1, \ldots, \mathbf{x}_N) = \frac{1}{\sqrt{N!}} \det \left[\psi_j(\mathbf{x}_k)\right]_{j,k}$$

Pauli's exclusion principle: $\boxed{\psi \wedge \psi = 0}$

Bose-Einstein statistics vs **Fermi-Dirac** statistics

(In 1D one has to care about boundary conditions at $r = 0$: $\partial_r \Psi = \eta\Psi$)

Quantum statistics: Spin-Statistics theorem

internal state $\mathfrak{h} \cong \mathbb{C}^D$

relativistic considerations $\Rightarrow \begin{cases} D \textbf{ odd} \text{ for bosons} \\ D \textbf{ even} \text{ for fermions} \end{cases}$

Here **spinless** particles ($D = 1$ and nonrel.)

[Doplicher, Haag, Roberts, 1971; Buchholz, Fredenhagen, 1982; Fröhlich, Gabbiani, Marchetti, 1989; Mund, 2009]

Commutators

Given an ON basis of $L^2(\mathscr{C}_1)$,

$$\{\psi_n\}_{n=0,1,2,\ldots},$$

we may define corresponding **annihilation / creation operators** on the Bose/Fermi part of the **Fock space** $\bigoplus_{N=0}^{\infty} \otimes^N L^2(\mathscr{C}_1)$:

$$a_n \colon \otimes_{\mathrm{sym}}^N L^2(\mathscr{C}_1) \to \otimes_{\mathrm{sym}}^{N-1} L^2(\mathscr{C}_1), \;\; a_n^* \colon \otimes_{\mathrm{sym}}^N L^2(\mathscr{C}_1) \to \otimes_{\mathrm{sym}}^{N+1} L^2(\mathscr{C}_1),$$

$$a_n a_m - a_m a_n = 0, \quad a_n a_m^* - a_m^* a_n = \delta_{nm}, \quad a_n^* a_m^* - a_m^* a_n^* = 0,$$

respectively

$$c_n \colon \wedge^N L^2(\mathscr{C}_1) \to \wedge^{N-1} L^2(\mathscr{C}_1), \;\; c_n^* \colon \wedge^N L^2(\mathscr{C}_1) \to \wedge^{N+1} L^2(\mathscr{C}_1),$$

$$c_n c_m + c_m c_n = 0, \quad c_n c_m^* + c_m^* c_n = \delta_{nm}, \quad c_n^* c_m^* - c_m^* c_n^* = 0.$$

Note $(c_n^*)^2 = 0$ (Pauli again).

BEC vs Fermi sea

The **non-interacting gas** of N particles in a box $\mathscr{C}_1 = \Omega$:

$$\hat{H}_N = \sum_{j=1}^{N} \hat{H}_1(\mathbf{x}_j) = \frac{\hbar^2}{2m} \sum_{j=1}^{N} (-\nabla^2_{\mathbf{x}_j})_\Omega,$$

with eigenstates

$$\Psi_{(n_j)} = \psi_{n_1} \otimes \ldots \otimes \psi_{n_N}, \qquad \hat{H}_1 \psi_n = e_n \psi_n, \qquad \sum_{j=1}^{N} e_{n_j}.$$

At zero temp. **bosons** form a **Bose-Einstein Condensate (BEC)**

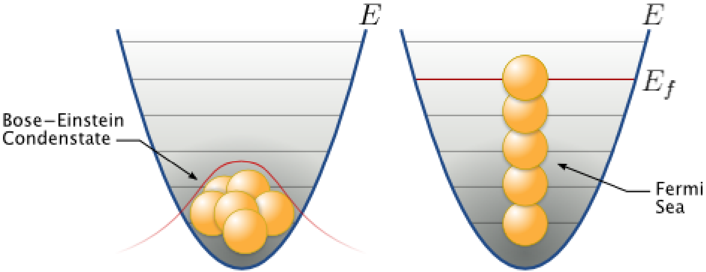
$$\Psi_{(0,\ldots,0)}(\mathbf{x}_1, \ldots, \mathbf{x}_N) = \psi_0(\mathbf{x}_1) \ldots \psi_0(\mathbf{x}_N), \quad N e_0,$$

while **fermions** fill a **Fermi sea** (lowest energy levels)

$$\Psi_{(0,1,\ldots,N-1)} = \psi_0 \wedge \psi_1 \wedge \ldots \wedge \psi_{N-1},$$

Weyl's law: $\sum_{n=0}^{N-1} e_n \sim C_d |\Omega|^{-2/d} N^{1+2/d}$ as $N \to \infty$.

BEC vs Fermi sea



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Density Functional Theories

One-body density $\varrho_\Psi \in L^1(\mathbb{R}^d;\mathbb{R}_+)$, $\int_{\mathbb{R}^d} \varrho_\Psi = N$ (Ψ normalized),

$$\varrho_\Psi(\mathbf{x}) := \sum_{j=1}^{N} \int_{\mathbb{R}^{d(N-1)}} |\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_{j-1}, \mathbf{x}, \mathbf{x}_{j+1}, \ldots, \mathbf{x}_N)|^2 \prod_{i \neq j} d\mathbf{x}_i$$

Trivially, for a one-body potential $V \colon \mathbb{R}^d \to \mathbb{R}$,

$$\langle \Psi, \textstyle\sum_{j=1}^N V(\mathbf{x}_j)\Psi \rangle = \int_{\mathbb{R}^d} V \varrho_\Psi.$$

Local Density Approximation (use Weyl in boxes locally):

$$\langle \Psi, \hat{H}_N \Psi \rangle \approx \int_{\mathbb{R}^d} \left(C_d \varrho_\Psi^{1+2/d} + V \varrho_\Psi \right) \quad \text{for minimizers.}$$

The r.h.s. is known as the **Thomas-Fermi (TF) functional**.

Important example: 2D external magnetic field

2D and **constant magnetic field** $B > 0$:

$$\mathscr{C}_1 = \mathbb{R}^2 \cong \mathbb{C}, \qquad z = \sqrt{\frac{B}{2\hbar}}(x + iy)$$

$$H_1 = \frac{1}{2m}\Big((p_x + By/2)^2 + (p_y - Bx/2)^2\Big)$$

$$L_1 = xp_y - yp_x = z\partial_z - \bar{z}\partial_{\bar{z}}$$

Landau level $n \in \{0, 1, 2, \ldots\}$, ang. mom. $l \in \{-n, -n+1, \ldots\}$

$$\hat{H}_1\psi_{n,l} = \frac{\hbar B}{m}\left(n + \frac{1}{2}\right)\psi_{n,l} \qquad \hat{L}_1\psi_{n,l} = \hbar l\psi_{n,l}$$

$$\psi_{0,l}(z) = \frac{1}{\sqrt{\pi l!}} z^l e^{-|z|^2/2}$$

N-body states

$$\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_N) = f(z_1, \ldots, z_N; \bar{z}_1, \ldots, \bar{z}_N) e^{-\sum_j |z_j|^2/2}$$

(Fractional) quantum Hall effect of N electrons:

$$H_N = \frac{1}{2m}\sum_{j=1}^{N}\Big((p_x - By/2)^2 + (p_y + Bx/2)^2\Big)_j + \sum_{j<k}|\mathbf{x}_j - \mathbf{x}_k|^{-1}$$

Laughlin's variational ansatz: $\Psi \sim \prod_{j<k} g(z_j - z_k)$ (Jastrow)

1. lowest Landau level $\Rightarrow \Psi \sim f(z_1, \ldots, z_N)e^{-|\mathrm{z}|^2/2}$
2. fermionic $\Rightarrow f$ antisymm. $\Rightarrow g$ odd
3. eigenstate of ang. mom. $\Rightarrow f$ homogeneous pol., $g(z) \sim z^\ell$

$\Rightarrow$

$$\Psi_{\mathrm{Lau}}(\mathrm{z}) = \prod_{j<k}(z_j - z_k)^\ell e^{-|\mathrm{z}|^2/2}, \qquad \ell \geq 1 \text{ odd}.$$

Coulomb gas (plasma) connection

$$|\Psi_{\mathrm{Lau}}(\mathrm{z})|^2 = \exp\Big(2\ell\sum_{j<k}\ln|z_j - z_k| - \sum_j |z_j|^2\Big)$$

2D clustering states

Laughlin **quasiholes**

$$\Psi_{\mathrm{qh}}(\mathrm{z}; w_1, w_2) = \prod_{j=1}^{N} (w_1 - z_j)^{\gamma_1} (w_2 - z_j)^{\gamma_2} \Psi_{\mathrm{Lau}}(\mathrm{z})$$

Pfaffian / Moore-Read states [Cappelli, Georgiev, Todorov]

$$\Psi(\mathrm{z}) = \mathcal{S}\left[\prod_{1 \le j < k \le N/2} (z_{1,j} - z_{1,k})^2 \prod_{1 \le j < k \le N/2} (z_{2,j} - z_{2,k})^2 \right] e^{-|\mathrm{z}|^2/2}$$

Read-Rezayi states

$$f_{N=\nu K}(\mathrm{z}) := \frac{1}{(\nu!)^{K-1}} \mathcal{S}\left[\prod_{q=1}^{\nu} \prod_{1 \le j < k \le K} (z_{q,j} - z_{q,k})^{\mu} \right], \quad \mu \text{ even}$$

Clustering property

$$f_N(z_1, \ldots, z_{N-\nu}, z, \ldots, z) = \prod_{j=1}^{N-\nu} (z - z_j)^{\mu} f_{N-\nu}(\mathrm{z})$$

Connections to CFT, Jack polynomials, ... [Bernevig, Haldane]

Further references

J. M. Leinaas, J. Myrheim, *On the theory of identical particles*, Nuovo Cimento B 37 (1977) 1.

D. L., N. Rougerie, *Emergence of fractional statistics for tracer particles in a Laughlin liquid*, Phys. Rev. Lett. 116 (2016) 170401, arXiv:1601.02508

D. L., *Many-anyon trial states*, Phys. Rev. A 96 (2017) 012116, arXiv:1608.05067

D. L., *Methods of modern mathematical physics: Uncertainty and exclusion principles in quantum mechanics*, lecture notes for a master-level course given at KTH in 2017 (revised May 2018), arXiv:1805.03063