FERMIONS AND BOSONS:
DETERMINANTS AND PERMANENTS

FROM PHYSICS
TO THE THEORY OF RANDOM POINT PROCESSES

February 6, 2019
Lille

Odile Macchi, Membre de l'Académie des sciences

OUTLINE

1. The context of research in physics around my thesis

2. The theory of coincidences

3. Photons and bosons in a chaotic state:
permanental point processes

4. Electrons and fermions in a chaotic state:
determinantal point processes

5. The symmetry between chaotic bosons and fermions

1. The context of research in physics around my thesis

1954 sq Studies of André Blanc-Lapierre about random fonctions

La Vie et l'Œuvre Scientifique d'André Blanc-Lapierre

Notice de Bernard Picinbono

La vie et l'œuvre scientifique d'André Blanc-Lapierre,
Membre de l'Académie des sciences
7 juillet 1915 - 14 décembre 2001
par Bernard Picinbono, Correspondant de l'Académie

André Blanc-Lapierre
a renown physicist and academician

- he left his mark by introducing in France the then emerging information theory,

- he extended it to optics,

- well ahead of his time, he seeked to immerse Statistical Mechanics in a rigorous probabilistic framework:

- Early in 1954 he first explained properly the phenomenom of optical coherence (interference fringes) with the theory of random functions and the notion of correlation. His papers, solely in French, were ignored (knowingly?)
- 10 years later, with the discovery and development of lasers, this viewpoint became universal. And it was extended by Roy Glauber in Quantum Mechanics.

1954 ABL explains optical coherence, alias correlation between light beam fluctuations

OPTIQUE. — *Sur la notion de cohérence en optique*. Note (*) de MM. **André Blanc-Lapierre** et **Pierre Dumontet**, présentée par M. Louis de Broglie.

Les auteurs précisent, dans le cas général de sources non monochromatiques, les notions de cohérence, de cohérence partielle et d'incohérence pour un ensemble de sources lumineuses. Des définitions mathématiques précises sont données dans le cadre de la théorie des fonctions aléatoires.

1. *Introduction*. — L'effet de la diffraction sur la correspondance objet-image a été surtout étudié dans les cas d'un *objet cohérent* ou d'un *objet incohérent* [1]. Récemment, certains auteurs [2] ont considéré le cas d'une *cohérence partielle*.

(*) Séance du 8 février 1954.

(1) P. M. Duffieux. *L'intégrale de Fourier et ses applications à l'Optique*, Rennes, 1946.
A. Blanc-Lapierre et M. Perrot, *Comptes rendus*, **231**, 1950, p. 539.

Applying to optics his new theory of random functions

Mathematical foundations:

- 'Theory of random functions' A. Blanc-Lapierre, R. Fortet, Ed. Masson, Vol. 1 1965, Vol. 2 1968

Optics:

- A light source creates an electromagnetic field $x(M,t)$ in space M at time t : it is a random function
- For incoherent light the field is a Gaussian random function

Coherence – Correlation:

- Two stationary light sources: in an observation point M two superimposed fields $x_1(t)$ and $x_2\,(t)$.
  To create interferences (fringes), put a delay τ on second source path $I(t,\tau) = [x_1(t) + x_2(t-\tau)]^2.$
- Average light intensity

$$J(\tau) = E\{\mathfrak{J}(t,\tau)\} = E\{x_1^2\} + E\{x_2^2\} + 2\,\gamma(\tau)$$

$$\gamma(\tau) = E\{x_1(t)\,x_2(t-\tau)\}$$

$$J(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_0^T x_1^2(t)\,dt + \lim_{T\to\infty}\frac{1}{T}\int_0^T x_2^2(t)\,dt + \\ + 2\lim_{T\to\infty}\frac{1}{T}\int_0^T x_1(t)\,x_2(t-\tau)\,dt,$$

At the same time, the first optical coherence (interference) experience at the quantum level: the photodetection times for the two light beams appear correlated: a bunching effect

1956
The HBT Experiment

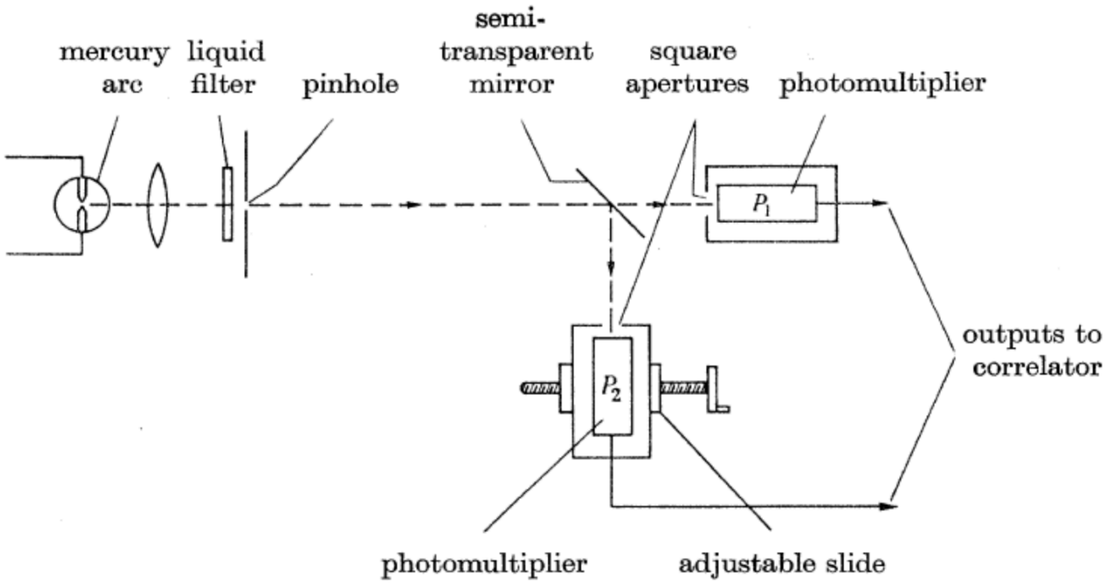

Fig. 3: The optical system set up by Hanbury Brown and Twiss.
Source: Hanbury Brown & Twiss, 1958, p. 299).

Reproduced from *Nature (London)* (1956) **177**, 27-32

CORRELATION BETWEEN PHOTONS IN TWO COHERENT BEAMS OF LIGHT

By *R.Hanbury Brown*
University of Manchester, Jodrell Bank Experimental Station
and
R.Q.Twiss
Services Electronics Research Laboratory, Baidock

“In this optical system, what is fundamental is that the time of arrival of photons at the two photocathodes should be correlated when the light beams incident upon the two mirrors are coherent. However, so far as we know, this fundamental effect has never directly been observed with light, and indeed its very existence has been questioned”

HB &T

They were right, this effect has never been observed with coherent light: it could not, since it occurs only with incoherent (chaotic) light. But we know in hindsight that the mercury light source they viewed as coherent was in fact incoherent, something they did not know and **could not know** because laser was not yet invented!

Bad early reception of the 1956 HBT experiment

This experiment favors the wave theory of light and seems to contradict quantum mechanics and the photon theory of light. It was very puzzling for physicists. In the fifties it provoked a heated debate about the concept of the photon.

What is easily conceivable in the classical wave formalism (e.g. with a radio wave) becomes hard to imagine at the quantum level: why should photons cling to each other?

Later (1991) Hanbury Brown wrote « To me the interesting thing about all this fuss was that so many physicists had failed to grasp how profoundly mysterious ligth really is, and were relectant to accept the practical consequences of the fact that modern physics does not claim to tell us what things are like 'in themselves' but only how they 'behave'… If our system was really going to work one would have to imagine photons hanging about, waiting for each other in space! »

In fact HB & T became the fathers of a new discipline « statistical optics » which investigates statistical laws for photoncounting under various physical situations:

« In my opinion HB & T is more a precursor of the quantum optics effects involving photon correlation » Alain Aspect, 2009

Some French research in physics at that time in Orsay

- 1958: Yves Rocard from Ecole Normale Supérieure who helped develop the French atomic bomb builds up in Orsay the linear accelerator of particles for fundamuntal research in physics
- 1961: André Blanc-Lapierre becomes the director of the accelerator and builds up a collision ring (Anneau de Collision –ACO)
- 1963: Collisions between electrons and positrons are observed at ACO for the first time in the world.
- 1965: Bernard Picinbono, a former doctoral student of A. Blanc-Lapierre (in Alger) arrives in Orsay, equipped with expertise in the necessary statistical tools: in optics, it is not possible to track the fluctuations of the EM field corresponding to a huge number of asynchronous light emitting atoms.
- 1966: Statistical optics launched in Orsay. Bernard had investigated the detection of weak optical signals, embedded in the backgroung random noise called 'shot noise' (bruit de grenaille). After A. Blanc-Lapierre, he deepens the statistics of fluctuating physical phenomena. He launches a new lab: 'Laboratoire d'études des phénomènes aléatoires' (LEPA), with research program in statistical optics and signal processing.
- LEPA was well located in the Institut d'electronique fondamentale (IEF) where experiences of optical illumination of particle and lasers were experimented with powerful electronic microscope designed!

Bernard Picinbono and statistical optics

- Bernard Picinbono wishes to understand why the bunching effect of photons disappears with monomode (coherent laser) light.
- He is convinced that the classical (ABL) approach works for laser as it does for incoherent (natural) light, and that it should explain why the bunching effect does not arise with coherent light
- In 1964 at the prestigious French École des Houches de Physique Théorique Bernard meets Roy Glauber who was presenting his quantum theory of optical coherence (2005 Nobel Prize)

After the laser discovery Roy Glauber introduces a new quantum formalism valid for coherent and incoherent beams: it explains optical coherence

VOLUME 10, NUMBER 3 PHYSICAL REVIEW LETTERS 1 FEBRUARY 1963

PHOTON CORRELATIONS*

Roy J. Glauber
Lyman Laboratory, Harvard University, Cambridge, Massachusetts
(Received 27 December 1962)

In 1956 Hanbury Brown and Twiss[1] reported that the photons of a light beam of narrow spectral width have a tendency to arrive in correlated pairs. We have developed general quantum mechanical methods for the investigation of such correlation effects and shall present here results for the distribution of the number of photons counted in an incoherent beam. The fact that photon correlations are enhanced by narrowing the spectral bandwidth has led to a prediction[2] of large-scale correlations to be observed in the beam of an optical maser. We shall indicate that this prediction is misleading and follows from an inappropriate model of the maser beam. In considering these problems we shall outline a method of describing the photon field which appears particularly well suited to the discussion of experiments performed with light beams, whether coherent or incoherent.

The correlations observed in the photoionization processes induced by a light beam were given a simple semiclassical explanation by Purcell,[3] who made use of the methods of microwave noise theory. More recently, a number of papers have been written examining the correlations in considerably greater detail. These papers[2,4-6] retain the assumption that the electric field in a light beam can be described as a classical Gaussian stochastic process. In actuality, the behavior of the photon field is considerably more

Bernard’s Orsay team of young scientists

1967 he hires a team of researchers to launch Statistical Optics in Orsay

experimenters in optics (lasers) :

Cherif Bendjaballah, Martine Rousseau-Le Berre

theoricians in mathematics and physics:

- *A mathematician for the bunching effect of bosons*:

  Odile Macchi, experienced in probabilities will deepen the therory of permanental processes

- *A physicist for fermions*:

  Christine Bénard, who chose the quantum formalism of Glauber

Both collaborate for fermions : ***discovery of determinantal processes***

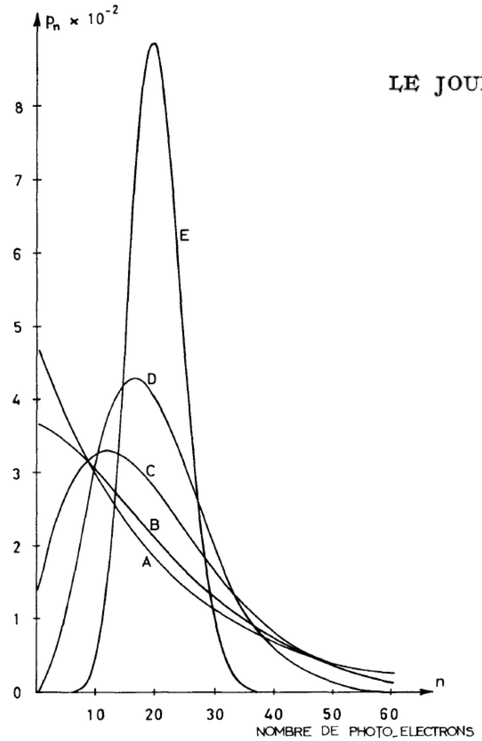

FIG. 1. — Distribution du nombre de photoélectrons dans une expérience en « relaxé », pour divers champs lumineux résultant de la superposition de champs gaussien et cohérent dans un seul mode.

L'intensité totale du champ est telle que le nombre moyen de photoélectrons comptés est 20 pour toutes les courbes.

La courbe A représente p_n pour un champ purement gaussien (éq. (4.2)).

La courbe E pour un champ cohérent (éq. (4.1)).

Les courbes B, C et D sont relatives aux champs superposés (éq. (4.3)), le paramètre m vaut respectivement 1/3, 1 et 3, d'après R. J. Glauber [24].

LE JOURNAL DE PHYSIQUE TOME 30, AOUT-SEPTEMBRE 1969, PAGE 675.

Clever experimental physicists

PROPRIÉTÉS STATISTIQUES DES PHOTOÉLECTRONS

Par MARTINE ROUSSEAU,
Institut d'Électronique Fondamentale [1], Faculté des Sciences, 91-Orsay.

(Reçu le 10 février 1969.)

A sound experimental study (photocounting and interval measurements) for the PP of electrons emitted by a cathode impinged on by various types of light beams, ranging from a chaotic beam (thermal light, curve A), unto a pure laser beam (coherent light, curve E), in connection with the fluctuation properties of the light intensity. The distribution of the number of electrons shows clearly that the bunching effect is reserved to chaotic beams

Adv. Appl. Prob. **7**, 83–122 (1975)
Printed in Israel

THE COINCIDENCE APPROACH TO STOCHASTIC POINT PROCESSES

ODILE MACCHI, *Laboratoire d'Etude des Phénomènes Aléatoires, Université de Paris-Sud*

Abstract

The structure of the probability space associated with a general point process, when regarded as a counting process, is reviewed using the coincidence formalism. The rest of the paper is devoted to the class of regular point processes for which all coincidence probabilities admit densities. It is shown that their distribution is completely specified by the system of coincidence densities. The specification formalism is stressed for 'completely' regular point processes. A construction theorem gives a characterization of the system of coincidence densities of such a process. It permits the study of most models of point processes. New results on the photon process, a particular type of conditioned Poisson process, are derived. New examples are exhibited, including the Gauss-Poisson process and the 'fermion' process that is suitable whenever the points are repulsive.

STOCHASTIC POINT PROCESS; JOINT COUNTING EVENT; COINCIDENCE PROBABILITY; COINCIDENCE DENSITY; EXCLUSION PROBABILITY DENSITY; REGULAR PROCESS; PHOTON PROCESS; GAUSS-POISSON PROCESS; FERMION PROCESS

1. Introduction

The most usual way to deal with stochastic point processes (P.P.) is to consider the distances or intervals between the successive points (or occurrences), a procedure which applies when the space is one-dimensional only. Then, in the classical approach [1]–[3], these quantities are taken as the basic random variables and are used to define the probability distribution of the P.P. In [4], the very similar idea of 'forward recurrence times' plays a crucial role in the study of P.P. However the interval-method cannot be generalized to multidimensional P. P.

The other relevant quantity is the counting process $N(t)$, i.e., the number of occurrences in the interval $(0,t]$. Obviously the concept of counting process can be extended to P.P. in a space of higher dimension. In fact, each realization of the counting process is a positive discrete measure on the space X where the occurrences are located. Putting a probability structure on the space of such 'counting' measures leads to possible definitions of P.P. such as given in [5]–[13]. Our work is in a close relationship with Moyal's work [5].

However Moyal defines the P.P. events and their probabilities by considering the points in space where the occurrences are located. He then shows ([5], Theo-

Received in revised form 9 September 1974.

THE FERMION PROCESS — A MODEL OF STOCHASTIC POINT PROCESS WITH REPULSIVE POINTS

ODILE MACCHI

ORSAY

1. INTRODUCTION

This paper presents a model of stochastic point process (P.P.) whose occurrences are repulsive to one another. The origin of the model arises in the theory of quantum particles and describes the statistical distribution of a fermion system in thermal equilibrium. For such a system, two P.P. can be defined which describe the places of the particles at a given time or the instants when any particle passes in a given point of space. Then physical arguments lead to the value of coincidence densities (C.D.) of these P.P.

In section 2, we briefly recall how the system of C.D. can define the statistics of a P.P. In section 3, we give the mathematical model of a fermion process (F.P.), starting from these C.D. and we point out repulsivity. Then we prove the existence of the F.P., give some details about the physical process, and illustrate the F.P. with an example.

2. PRELIMINARIES: REGULAR POINT PROCESSES

For the sake of clarity, we must recall some useful material about P.P. These ideas can be found with more details in [1], particularly a definition of P.P. suitable for our purposes. We consider P.P. in $\mathbf{R}^M$ that is, the state space for the occurrences is a Borel-set X in $\mathbf{R}^M$. The reference measure is the usual Lebesgue-measure.

2.1. Exlusion Probability Densities

Let $p_n(t_1, \ldots, t_n)$, $n = 0, 1, \ldots$, be a sequence of non-negative, symmetrical and measurable functions respectively defined on X^n.

J. Koževnik (ed.), *Transactions of the Seventh Prague Conference on Information Theory, Statistical Decision Functions, Random Processes and of the 1974 European Meeting of Statisticians*

2014: a German mathematician writes a strikingly humble letter in French to Odile Macchi

‘Madame, m’autorisez-vous à traduire votre thèse française (200 pages) pour publication sous forme de livre anglais par mon ami, l’éditeur allemand Dr. Walter Warmuth?’ Professor Hans Zessin, Berlin

in 2017: the book in English
Hans: Why hundreds of non rewarding hours for such a work?
Walter: Why taking the risk of editing a book with a tiny market and no profit?

Merely for the sake of making truth known better!

‘To give this seminal work the place it deserves, as a cornerstone which connects Quantum Optics and modern point process theory’
(Book introduction p. 7)

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présentée au

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DOCTEUR ÈS SCIENCES PHYSIQUES

PROCESSUS PONCTUELS ET COINCIDENCES

Contribution à l'étude théorique des processus ponctuels

Applications à l'Optique Statistique et aux Communications Optiques.

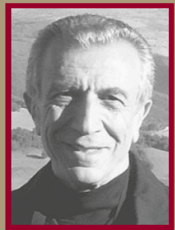
par

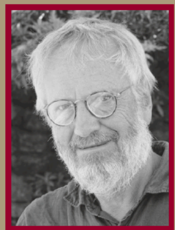
Odile MACCHI née DANJOU

Soutenue le Janvier 1972 devant la Commission d'Examen.

Jury : A. BLANC-LAPIERRE président

R. FORTET
B. PICINBONO
P.L. HENNEQUIN
examinateurs





Odile Macchi. Directeur de Recherche émérite au Laboratoire des Signaux et Systemes du CNRS. Docteur d'état ès Sciences Physiques in 1972. Elected Correspondant in 1994 and Member in 2004 of the Académie des Sciences in the section Sciences mécaniques et informatiques. She received the Monpetit award of the French Academy of Sciences in 1991 and the European Technical Achievement Award of the European Association for Signal Processing in 1999. Among the numerous articles the most representative have their roots in the present work. She obtained the title Officier de l'Ordre national du mérite in 2008. She participated in numerous missions of humanitary character in India and certain african countries, some of them during a war. She wrote four books, in particular Adaptive processing: The Least Mean Square Approach with Applications in Communications, Wiley 1995.

Suren Poghosyan is Leading Researcher at the Institute of Mathematics of the Armenian National Academy of Sciences. He obtained the degree Sci.D (Full Doctor of Physical and Mathematical Sciences) from the Steklov Mathematical Institute, St. Petersburg branch in 2013.

Hans Zessin was professor at the Universities of Lille 1 (France) and Bielefeld (Germany). He retired in 2010.

This work contains the fundamental discovery of permanental processes as a model for photons made by Prof. Odile Macchi at the beginning of the seventies. It is deeply rooted in Quantum Optics in taking the notion of coincidence as a basic theoretical notion and develops for the first time in a rigorous way a point process theory for the description of bosons and fermions. It is a corner stone which connects Quantum Optics and modern point process theory. The purpose of the additional Scholion, written by Prof. Suren Poghosyan and Prof. Hans Zessin is an exposition of developments in the last four decades.

Classical Lectures

Editor: Mathias Rafler

Odile Macchi Point Processes and Coincidences

Odile Macchi

Point Processes and Coincidences

Contributions to the Theory. With Applications to Statistical Optics and Optical Communication

Classical Lectures

Walter Warmuth Verlag

Conclusion … of my thesis (January 12, 1972)

My viewpoint: find theoretical tools matched to physics and experimentation

Goal: use these tools to deepen the statistical study of bosons and fermions

Further studies and generalizations: this is up to … YOU!

- apply such tools to other problems and in particular in the theoretical statistics: - find new PP models thanks to the theory of coincidences

- enlarge the theory to
  - abstract spaces
  - marked or other more general PP

2. The theory of coincidences:
Indistinguishable occurrences
Validity in $\boldsymbol{\mathcal{X}} \subset \mathbb{R}^{\boldsymbol{n}}$
No assumption of stationarity

The coincidence densities

$$h_k(t_1,\ldots,t_k)\,dt_1 \ldots dt_k = \mathrm{E}[dN(t_1)\ldots dN(t_k)] = \Pr[dN(t_1) = \ldots = dN(t_k) = 1]$$

The h_k are the probability densities that the PP has k occurrences located in the infinitesimal intervals $\{t_j, dt_j\}$, $j = 1, \ldots, k$, **no matter** whether there are other occurrences elsewhere or not

The CD are the limiting aspect of the joint counting distributions. They are **physically measurable local quantities, independent of the volume $\mathcal{X}$** where the PP is observed. Thus they allow extending the definition of the PP over *all* $\mathbb{R}^n$.

They are non negative, bounded, symmetrical functions.

They are also the densities of the factorial moments of the numbers of points in disjoint intervals.

Not any sequence of non negative symmetrical bounded functions (even properly normalized) represents the CD of a PP. **There are conditions.**

For a **non stationary Poisson process** with intensity $\rho(t)$, the $dN(t_i)$are independent:

- $h_k(t_1,\ldots,t_k) = \rho(t_1)\ \ldots\ \rho(t_k)$.

The exclusion densities

$$p_n(t_1, \ldots, t_n) = n!\,\Pr[N(X) = n, dN(t_1) = \ \ldots = dN(t_n) = 1]$$

The p_n are the probability densities that the PP has **exactly *n* occurrences** located in the infinitesimal intervals $\{t_j, dt_j\}$, $j = 1, \ldots, n$, **exclusive** of any other occurrence elsewhere

Any sequence of non negative bounded symmetrical functions

$p_n(t_1, \ldots, t_n)$ normalized according to

$$\sum_{n=0}^{\infty} \frac{1}{n!} \int_{X^n} p_n(t_1, \cdots, t_n)\, dt_1 \cdots dt_n = 1 .$$

defines a unique regular PP having this sequence as ED. **No other condition.**

However **the ED are not physically measurable**

For a Poisson process with intensity $\rho(t)$

- $p_n(t_1, \ldots, t_n) = \rho(t_1) \ \ldots \ \rho(t_n)\, e^{-\int_X \rho(\theta) d\theta}$

Important quantities

The generating function of the total number N of occurrences in $\boldsymbol{\mathcal{X}}$

$g\,(v) = \mathrm{E}\,\{(1\ - v)^N\}$

The probability of no occurrence in the observed volume $\mathcal{X}$ is

$\mathrm{P}\,\{N\ = 0\} = g\,(1)$

$g\,(v)$ expands $\sum_{p=0}^{\infty}(-v)^p\; E\,(N^{[p]})/\mathrm{p}!$) which generates the factorial moments,

$E\,(N^{[p]}) = E\,(N(N-1)\ldots(N-p+1))$

The fact that $g\,(v)$ is the generating function of a non negative integer plays a basic role in the theory

The joint counting factorial moments in q disjoint subsets I_1 , I_2 , … , I_q of $\mathcal{X}$

$M\,\{p_q\} = \mathrm{E}\,\{N_1^{[p_1]}\ldots N_q^{[p_q]}\}$ are obtained by mere integration of $h_{p_1+\,\ldots+p_q}$ over I_1, I_2, … I_q

Regular PP

Let $\boldsymbol{a_p}$ be the convergence radius of the entire series associated to the system of ED

$$\Phi_p(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{X^n} p_n(t_1, \cdots, t_n) dt_1 \cdots dt_n \quad (= \sum_n z^n \ \mathrm{P}\{N(X) = n\})$$

Result 1

if $\boldsymbol{a_p} > 1$ the probability law of the PP is completely characterized equivalently by the system of CD or by the system of ED. Then the system of CD follows from the system of ED through **the direct relationships**

$$h_k(t_1, \cdots, t_k) = \sum_{j=0}^{\infty} \frac{1}{j!} \int_{X^j} p_{k+j}(t_1, \cdots, t_k, \theta_1, \cdots, \theta_j)\, d\theta_1 \cdots d\theta_j,$$

and the PP is called regular

Moreover $\boldsymbol{a_h = a_p}$ - 1, where $\boldsymbol{a_h}$ is the convergence radius of the entire series $\Phi_h(z)$ similarly associated to the system $h_k(t_1, \ldots, t_k)$ of CD

Completely regular PP

A regular PP is said completely regular if its system of ED can be derived from its system of CD through the **inverse relationships**

$$p_n(t_1, \cdots, t_n) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \int_{X^j} h_{n+j}(t_1, \cdots, t_n, \theta_1, \cdots, \theta_j)\, d\theta_1 \cdots d\theta_j$$

Result 2: if $\boldsymbol{a_p > 2}$ or equivalently $\boldsymbol{a_h > 1}$ the PP is completely regular.

Result 3 (basic) : Coincidence based construction of a completely regular PP

A system of non negative, bounded, symmetrical functions $\{h_k\,(t_1, \dots, t_k)\}$ is the system of CD of a completely regular PP if $\boldsymbol{a_h > 1}$ and if all the **series** $\boldsymbol{p_n(t_1, \dots, t_n)}$ **are non negative** (the normalizing condition is automatically satisfied). Unicity of this PP

Result 4: If $\boldsymbol{a_h} < 1$ the PP is not completely regular, one cannot derive the ED from the CD. There is no general result if $\boldsymbol{a_h} = 1$.

3. Photons and bosons in a chaotic state: permanental point processes

Cox (or Conditioned Poisson) Processes: general regular case

Definition: A regular Cox process is a PP whose CD are the moments of some non negative function $I(t)$ called the underlying intensity:

$$h_k(t_1, \ldots, t_k) = \mathrm{E}\,[I(t_1)\ \ldots I(t_k)]$$

Let I be the **bounded** rectangle in $\mathbb{R}^L$ where the PP is observed, and $I(t)$ a non negative random function. Then, given a sample of $I(t)$, the PP is Poisson with non stationary intensity $I(t)$, because the form $I(t_1)\ \ldots I(t_k)$ of its conditional CD is characterisitic of a Poisson process. Thus physicists call this PP a conditioned Poisson process.

Result 5 Existence theorem: If a non negative random function $I(t)$ is such that

$$\mathrm{E}\left[\exp a \int_{\mathrm{I}} I(\theta)d\theta\right] < \infty, \quad \text{for some } a > 0,$$

there exists a unique regular Cox Process with underlying intensity $I(t)$ on I. Its CD are as above and its ED are

$$p_n(t_1, \ldots, t_n) = \mathrm{E}\left[I(t_1)\ \ldots I(t_n) \exp a \int_{\mathrm{I}} I(\theta)d\theta\right]$$

If moreover $a > 1$, this Cox process is completely regular.

Result 6: We have proved the existence of Cox processes whose underlying intensities $I(t)$ assume sometimes negative values. This is possible e.g. with a Gaussian $I(t)$ whose (positive) mean value $m(t)$ dominates the (positive) covariance function in the sense that $m(t) \geq \int_I C(t,u)du$ a.e.

1965: Detecting photons of a light beam
a general Cox Process

Rev. Mod. Phys. 37, pp. 231-287, **1965** "*Coherence properties of optical fields Reviews of Modern Physics*" **Mandel L. and Wolf E.:** with both the classical and quantum mechanics formalisms, they showed that at the output of a noiseless photodetecting surface impinged on at ($t = \{\text{time } \theta, \text{place } x\}$) by a partially polarized weak beam of light (natural or laser), the PP is a Cox process whose underlying intensity is the (random) light intensity :

$$h_k(t_1, \ldots, t_k) = \mathrm{E}\,[I(t_1)\ \ldots I(t_k)];\ p_n(t_1, \ldots, t_n) = \mathrm{E}\Big[I(t_1)\ \ldots I(t_n)\exp \boldsymbol{s} \int_{\mathrm{I}} I(\theta)d\theta\ \Big]$$

where s is the detector efficiency, I is the bounded rectangle in $\mathbb{R}^4$ where the PP is observed, ***X*(*t*) the complex analytical signal associated to the real electromagnetic field *E*(*t*); and $I(t) = s\,|X(t)|^2$**

There are different Cox processes for various statistical properties of $E(t)$ (thus of $I(t)$).

Cf in particular the many papers of **B. Picinbono's team in the years 1968- 1975.**

Perfect laser: only the phase of $E(t)$ fluctuates, $I(t)$ non random, the PP is nonstationary Poisson.

1970 - Christine Bénard's model: Beams of quantum particles with the wave packet formalism

Phys. Rev A, vol 2, n° 5, 2140-2153, Nov. **1970** *« Fluctuations of Beams of quantum particles »*

Extending the work done at second order by M.L. Goldberger, 1963, she considered the coincidences of all orders: **in a finite rectangle cavity I, a random number *N* of noninteracting and indistinguishable particles** $\{\boldsymbol{Q_i}\}$**,** either bosons or fermions are superimposed. $\{Q_i\}$ is described by its **random 'wave packet'** $\emptyset(\boldsymbol{t_i})$, $t_i = \{\text{ time } \theta_i, \text{location } xi\}$. This is an unobservable model, no detetion considered.

The wave packets are independent of one another and have covariance function denoted $\boldsymbol{f}(\boldsymbol{t},\boldsymbol{u}) = E\{\emptyset(t)\ \emptyset^*(u)\}$. Means are quantum mechanical averages.

The beam (EM field) is described by its **random 'wave function', superposing all wave packets** by appropriate projections

☼ *on a symmetrized space for bosons* (integer spin)

☼ *on an anti-symmetrized space for fermions* (spin = odd half-integer).

The EM field covariance, ***C(t, u)***, follows from the wave packets covariance

But the resut is intractable unless the beam has weak density

Even then it is neat for bosons, but not for fermions.

The permanental model for a chaotic beam of bosons e.g. photons of a thermal light

Result 7: For chaotic bosons the CD are written solely with **the covariance of the wave function** $C\,(t_i, t_j) = \; E\left[X(t_i)\; X^*(t_j)\right]$ (not necessarily stationary). **They are the permanents**

$$h_k(t_1\,,\dots,\,t_k) = \sum P_\alpha \;\; \prod_{i=1}^{k} C(t_i \;\;, t_{\alpha_i}) \;\; \text{of the matrices} \qquad \begin{pmatrix} C(t_1,t_1) & \cdots & C(t_1,t_k) \\ \vdots & \ddots & \vdots \\ C(t_k,t_1) & \cdots & C(t_k,t_k) \end{pmatrix}$$

$\sum P_\alpha$ means summing over all permutations $(\alpha_1, \dots \alpha_k)$ of $(1, \dots, k)$

Why? Because the field $E(t)$ is a zero-mean Gaussian random process. Its analytical signal $X(t)$ (no negative frequencies) is a complex, zero mean, strongly Gaussian process with $\mathrm{E}\left[X(t_i)\;\mathrm{X}(t_j)\right] = 0$.

Result 8 (existence): There exists a (unique) PP with the above CD
The proof uses $\{\varphi_i(t)\}$, a complete system of orthonormal functions on I, that are eigenfunctions of $C(t, u)$

$$\lambda_i\,\varphi_i(t) = \int_{\mathrm{I}} \;\; C(t,u)\;\varphi_i(u)\;du \qquad \lambda_i > 0, \qquad \sum_i \lambda_i < \infty, \qquad \int_{\mathrm{I}} \;\; \varphi_i(t)\varphi_j^*(t)\;\mathrm{dt} = \delta_{i,j}$$

and the Kahrunen-Loeve expansion $C(t, u) = \sum_i \lambda_i\;\varphi_i(t)\varphi_i^*(u)$,
to show positivity of the CD, and validity of condition $\mathrm{E}\left[\;\exp a \int_{\mathrm{I}} \;\; I(\theta)d\theta\;\right] < \infty$, e.g. for $a < 1\;/(2\;\lambda_{max})$

The exclusion densities themselves are permanents

Result 9 The generating function for the detection of a chaotic boson beam is

$$\mathrm{E}\,\{(1 - v)^N\} = g(v) = \prod_{i=1}^{\infty} 1/(1 + vs\lambda_i)$$

With the help of the function $f(t,u)$ **related to the field covariance *C(t, u)*** through

$$f(t,u) + s\int_I C(t,\theta) f(\theta,u)\,\mathrm{d}\theta = C(t,u)$$

and with the permanents of the matrices $\begin{pmatrix} f(t_1,t_1) & \cdots & f(t_1,t_k) \\ \vdots & \ddots & \vdots \\ f(t_k,t_1) & \cdots & \mathrm{f}(t_k,t_k) \end{pmatrix}$

Then the ED are $\quad p_n(t_1, \ldots, t_n) = s^n\, h(s) \sum P_\alpha \prod_{i=1}^{n} f(t_i\ , t_{\alpha_i})\ h\,(\mathrm{s}) = g\,(1)$

Moreover $\quad f(t, u) = \sum_i (\lambda_i\,/(1 + s\lambda_i))\ \varphi_i(t)\varphi_i^*(u)$

which evidences that *f(t, u)* is a covariance function and thus that **these ED are indeed positive**

Physically it turns out that ***f(t, u)*** $= \boldsymbol{E}\{\emptyset(\boldsymbol{t})\ \emptyset^*(\boldsymbol{u})\}$ **is the wave packets' covariance**

Consistency

***C(t,u)* is the covariance function of the wave function,** *X(t)*, the complex random electromagnetic field

Thus *C(t,u)* is positive definite

This fact is of critical importance for the consistency of the PP model.

Through the positivity of the eigenvalues λ_i, this fact implies that *f(t,u)* is also positive definite.

In turn positive definiteness of *f(t,u)* controls the positivity of the function *h(s)* and of the permanental expressions of the CD and ED.

Positive definiteness of *f(t,u)* also controls the (existence) sufficient condition $\mathrm{E}\left[\exp^{a \int_{\mathrm{I}} I(\theta)d\theta}\right] < \infty$ for some $a > 0$.

Physical considerations show that ***f(t,u)* is the covariance function of the wave packets**

The bunching effect of the permanental PP

.

Result 10: Bunching effect :

$$h_2(t_1, t_2) = h_1(t_1)\, h_1(t_2) + s^2\, |C(t_1, t_2)|^2 > h_1(t_1)\, h_1(t_2)$$

Or equivalently

$$\Pr\{\, dN(t_2) = 1 / dN(t_1) = 1\} = \Pr\{\, dN(t_2) = 1\} + s\, |C(t_1, t_2)|^2 / C(t_1, t_1)$$

The *a posteriori* probability to detect any photon at a given time, when another photon has been detected in a neighboring time (second order coincidence) is higher than the *a priori* probability to detect a photon at that time: two photons tend to agregate!

Chaotic photons and bosons behave like sheeps!

Stationary (time) case : $C(t+d,t) = \Gamma(d)$

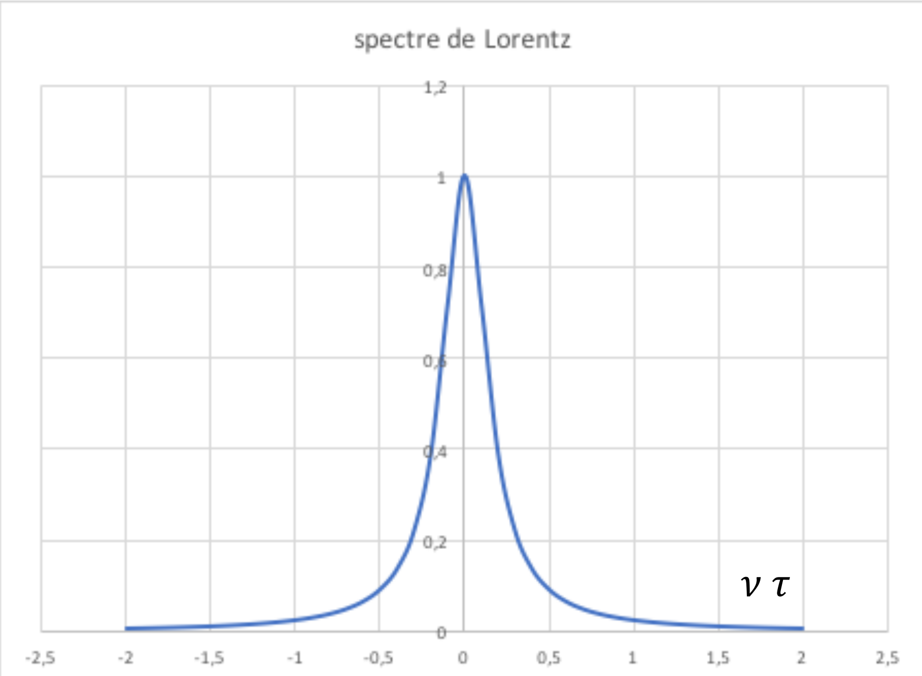
$E(N) = T\,\bar{I}$

Lorentz spectrum: τ = coherence time of the light field

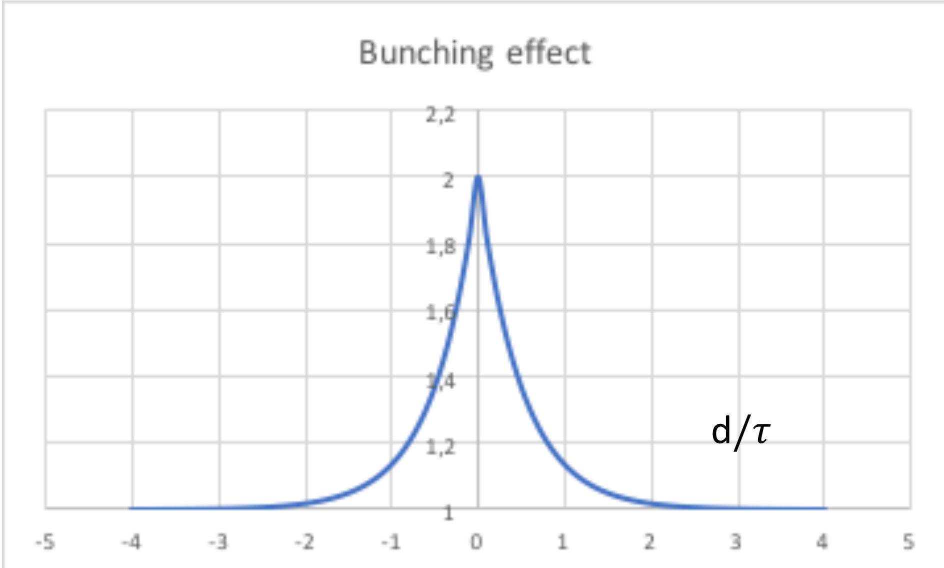
$\Gamma(d) = \bar{I}\exp(-|d|/\tau)$

Bunching effect: $B_2(d) = \boldsymbol{h_2(t_1, t_2)/\, h_1(t_1)\, h_1(t_2)}$

$= 1$ $\color{red}{\boldsymbol{+\exp(-2|d|/\tau)}}$ (1: no agregation)



$\boxed{\gamma(\nu) = 2\,\bar{I}\,\tau\,/(1 + 4\,\pi^2\tau^2\nu^2)}$



Higher order bunching effects of photons (Lorentzian spectrum light)

Three points t_1 ; $t_2 = t_1 + d_1$; $t_3 = t_2 + d_2$, $d_1 > 0$; $d_2 > 0$

If $B_3(d_1, d_2) = h_2(t_1, t_2, t_3) / h_1(t_1)\, h_1(t_2)\, h_1(t_3)$

$B_3(d_1, d_2) = B_2(d_1) + B_2(d_2) + B_2(d_1 + d_2) +$ **2 exp – 2 $(d_1 + d_2)/\tau$)**

The bunching factor between three occurrences is superior to the sum of the three pairwise bunching factors

The agregation tendency is reinforced as the number of occurrences increases.

4. Electrons and fermions in a chaotic state: determinantal point processes

A very long germination …

Feb. 22, 2013

Etienne Ghys to Odile:

On this board, is it you « Théorème de Macchi 1975 » ?

Odile to Etienne

« Certainly not! »

Here Alexander Bufetov, of the Lyon 1 University, at an LATP 2013 Colloquium like the one of today

1973: The determinantal model for a beam of fermions

Again Christine Bénard, but with Odile Macchi: J. Math. Ph., vol 14, n° 2, 155-167, Feb. **1973** « Detection and 'emission' processes of quantum particles in a chaotic state». With the wave packet formalism we considered **a random number N of noninteracting and indistinguishable fermions** (spin = odd half-integers) in a rectangle cavity: electrons, protons, neutrons …

The particle $\{Q_i\}$ found in t_i is described by its **random wave packet** $\emptyset(t_i)$, not necessarily real, but independent of the other wave packets. The **random wave function of the fermion beam** follows by appropriate projection on an anti-symmetrized space. Means are quantum mechanical averages with respect to the field operator. Then the very intricate expression of the CD involves the **covariance *C(t, u)* of the wave function.**

For a chaotic beam, the CD reduce to the determinants of the covariance matrix

$$h_k(t_1, \dots, t_k) = \det C\{t_k\} = \sum P_\alpha \ (-1) \prod_{i=1}^{k} C(t_i \ , t_{\alpha_i})$$

$$\text{where} \quad C\{t_k\} = \begin{pmatrix} C(t_1, t_1) & \cdots & C(t_1, t_k) \\ \vdots & \ddots & \vdots \\ C(t_k, t_1) & \cdots & C(t_k, t_k) \end{pmatrix}$$

$\sum P_\alpha(-1)$ means summing over all permutations $(\alpha_1, \dots \alpha_k)$ of $(1, \dots, k)$, each term affected with the sign $(-1)^{r(\alpha)}$, $r(\alpha)$ denoting the sign of the permutation $(\alpha_1, \dots \alpha_k)$.

Existence of determinantal PP

Result 11, existence of DPP: The necessary and sufficient conditions for a series of bounded, symmetrical, non negative functions $h_k(t_1 ,..., t_k)$ of the determinantal form to be the CD of a regular PP on I are

Condition 1. The function ***C(t, u)*** on which the functions $h_k(t_1 ,..., t_k)$ are based is **positive definite**.

This condition is necessary and sufficient for non negativity of all the $h_k(t_1 ,..., t_k)$

Condition 2. $\boldsymbol{\lambda_i < 1}$ **for all *i*.**

Under condition 1 there exists a complete system of orthonormal functions $\{\varphi_i(t)\}$ on I that are eigenfunctions of *C(t, u)*

$$\lambda_i \varphi_i(t) = \int_{\mathrm{I}} C(t,u)\, \varphi_i(u)\, du \quad \int_{\mathrm{I}} \varphi_i(t)\varphi_j^*(t)\, dt = \delta_{i,j},$$

and such that $$C(t, u) = \sum_i \lambda_i\, \varphi_i(t)\varphi_i^*(u),$$

with the basic properties that $\boldsymbol{\lambda_i > 0}$, $\sum_i \boldsymbol{\lambda_i} < \infty$

Proof of condition 2

A necessary condition: If the model is consistent

$$g(v) = \prod_{i=1}^{\infty} (1 - v\lambda_i)$$

is the generating function of the non negative, integer number N of fermions

This requires that $\boldsymbol{\lambda_i \leq 1, \forall\, i}$

and even that $\boldsymbol{\lambda_i < 1, \forall\, i}$

($\lambda_{i_0} = 1$ for some i_0 would yield all the ED = 0)

A sufficient condition: Assume that $\boldsymbol{\lambda_i < 1, \forall\, i}$

Then the inversion formalism is valid and yields the functions

$$p_n(t_1, \dots, t_n) = \prod_{i=1}^{\infty} (1 - \lambda_i) \sum P_\alpha \, (-1) \prod_{i=1}^{n} f(t_i \;, t_{\alpha_i})$$

where $f\,(t, u)$ is the resolvent of the Fredholm equation:

$$f\,(t, u) - \int_{\mathrm{I}} f\,(t, \theta) C(\theta, u) d\theta = C(t, u) \quad t, u \in \mathrm{I}$$

It is worth $f\,(t, u) = \sum_i (\lambda_i / (1 - \lambda_i))\, \varphi_i(t) \varphi_i^*(u)$, and thus is positive definite.

Therefore the $p_n(t_1, \dots, t_n)$ are non negative, they are indeed the ED

Physical interpretation: the Pauli exclusion principle

In the wave packet formalism, denote $\psi_k\ (t)$ a complete system of orthonormal modes for the (bounded) cavity I, n_k the number of fermions in mode k, $\langle n_k \rangle$ its average.

Physical considerations show that the wave packet covariance $g(t, u) = \langle \emptyset(t) | \emptyset(u) \rangle$reads:

$$g(t, u) = \sum_k \left(\langle n_k \rangle / (1 - \langle n_k \rangle) \right) \psi_k\ (t)\ \psi_k^*(u)$$

Identification with our model Let $\lambda_k = \langle n_k \rangle$ be the average numbers of fermions per mode

$\varphi_k(t) = \psi_k\ (t)$ be the modes of the cavity I

then our $f(t, u) = \sum_i\ (\lambda_i / (1 - \lambda_i))\ \varphi_i(t) \varphi_i^*(u)$ is the wave packet covariance

Clearly the λ_i must be positive (condition 1): they are the mean number of fermions per mode

And the λ_i should be less than 1 (condition 2): this is a property specific of fermion beams:

'At most one fermion per quantum mode', i.e. the Pauli exclusion principle

$C(t, u) = \sum_k \lambda_k\ \varphi_k(t) \varphi_k^*(u)$ becomes the (quantum) covariance of the wave function (field).

The anti-bunching effect of the determinantal PP

Result 12: Anti-bunching effect :

$$h_2(t_1, t_2) = h_1(t_1)\, h_1(t_2) - |C(t_1, t_2)|^2$$

$$B_2(t_1, t_2) = h_2(t_1, t_2) / h_1(t_1)\, h_1(t_2) = 1 - |C(t_1, t_2)|^2 / (h_1(t_1)\, h_1(t_2)) < 1$$

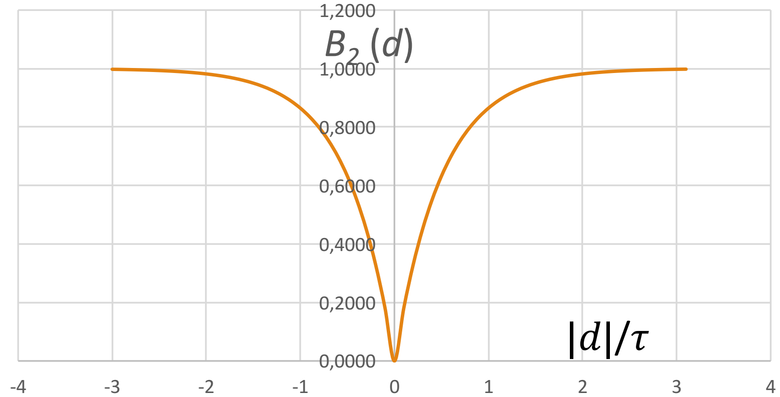
(1: no exclusion)

$B_2(t, t) = 0$ **absolute exclusion**

The *a posteriori* probability to detect any fermion at a given time, when another fermion has been detected in a neighboring time (second order coincidence) is smaller than the *a priori* probability to detect a fermion at that time: two fermions tend to exclude each other! At higher orders the determinantal expressions of CD exhibit similar exclusion properties.

Chaotic fermions (electrons) behave like foes!

Example: The time DPP with Lorentzian properties



1. The stationary case: Lorentz spectrum

$C(t+d, t) = \Gamma(d) = \bar{I} \exp(-|d|/\tau):$

$B_2(d) = 1$ **$- \exp(-2|d|/\tau)$**

2. The non stationary case: generalized renewal PP

$C(t_1, t_2)\, C(t_2, t_3) = C(t_1, t_3)\, C(t_2, t_2)$ with $t_1 \leq t_2 \leq t_3$

Let $D(t, u)$ be the normalized wave covariance: $C(t, u) = D(t, u)\sqrt{C(t,t)C(u,u)}$

$h_n(t_1, \ldots, t_n) = \prod_{i=1}^{n} C(t_i, t_i) \sum P_\alpha\ (-1) \prod_{i=1}^{n} D(t_i\ , t_{\alpha_i})$

$h_n(t_1, \ldots, t_n)\ = \prod_{i=1}^{n-1}\ (1\text{-}|D(t_i, t_{i+1})|^2) \prod_{i=1}^{n} h_1(t_i\ )$ with **successive times $t_1 \leq \ldots \leq t_3$**

This characterizes generalized renewal:

intervals between successive occurrences are independent but not equidistributed

5. The symmetry between chaotic bosons and fermions

The general inversion formlism	
$h_k(t_1,\cdots,t_k) = \sum_{j=0}^{\infty} \frac{1}{j!} \int_{X^j} p_{k+j}(t_1,\cdots,t_k,\theta_1,\cdots,\theta_j)\, d\theta_1 \cdots d\theta_j,$	$p_n(t_1,\cdots,t_n) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \int_{X^j} h_{n+j}(t_1,\cdots,t_n,\theta_1,\cdots,\theta_j)\, d\theta_1 \cdots d\theta_j$
The chaotic fields	
$f(t,u) = E\{\emptyset(t)\ \emptyset^*(u)\}$ is the wave packets' covariance	
$C(t,u) = \sum_i \lambda i\ \varphi i(t)\varphi_i^*(u)$ is the beam covariance, $\lambda i > 0,\ \sum_i \lambda i < \infty$	
$\lambda i\ \varphi i(t) = \int_I C(t,u)\ \varphi i(u)\ du \quad \int_I \varphi i(t)\varphi_j^*(t)\ dt = \delta_{i,j},$	
Bosons	Fermions
$\lambda i > 0$	$0 < \lambda i < 1$
$g(v) = \prod_{i=1}^{\infty} 1/(1+sv\lambda i),\ h(s) = g(1)$	$g(v) = \prod_{i=1}^{\infty}(1 - v\lambda i)$
$f(t,u) + s\int_I C(t,\theta) f(\theta,u)\,\mathrm{d}\theta = C(t,u)$	$f(t,u) - \int_I f(t,\theta)C(\theta,u)d\theta = C(t,u) \quad t,u \in \mathrm{I}$
$f(t,u) = \sum_i(\lambda i\ /(1+s\lambda i))\ \varphi i(t)\varphi_i^*(u)$	$f(t,u) = \sum_i\ (\lambda i\ /(1-\lambda i))\ \varphi i(t)\varphi_i^*(u)$
Permanents $h_k(t_1,\ldots,t_k) = \sum P_\alpha\ \prod_{i=1}^{k} C(t_i\ ,t_{\alpha i})$	Determinants $h_k(t_1,\ldots,t_k) = \sum P_\alpha\ (-1) \prod_{i=1}^{k} C(t_i\ ,t_{\alpha i})$
$p_n(t_1,\ldots,t_n) = s^n\ h(s) \sum P_\alpha\ \prod_{i=1}^{n} f(t_i\ ,t_{\alpha i})$	$p_n(t_1,\ldots,t_n) = \prod_{i=1}^{\infty}\ (1-\lambda i) \sum P_\alpha\ (-1) \prod_{i=1}^{n} f(t_i\ ,t_{\alpha i})$
Bunching effect : $\boldsymbol{h_2}(t_1,t_2) = \boldsymbol{h_1}(t_1)\ \boldsymbol{h_1}(t_2) + s^2\ \lvert \boldsymbol{C}(t_1,t_2)\rvert^2$	Anti-bunching effect $\boldsymbol{h_2}(t_1,t_2) = \boldsymbol{h_1}(t_1)\ \boldsymbol{h_1}(t_2) - \lvert \boldsymbol{C}(t_1,t_2)\rvert^2$

Which experimental results?

Bosons: At the epoch we wrote our paper (1969-1972) the laser was invented and much could be done experimentally to evidence the bunching effet. Other bunching effects have been observed with lasers, according to which part of the field $E(t)$ fluctuates (e.g. only the phase, then $I(t)$ non random, the PP is nonstationary Poisson).

How is the bunching effect observed now? Has a bunching effect higher than for chaotic photons been observed ? What about boson particles other than photons?

Fermions: At this epoch all experimental fermion sources, even the best monocinetic and powerful ones (point-cathode electron sources) could not provide coherence times larger than 10^{-13} sec., while electronic detection devices involved integration over times on the order of 10^{-9} sec. much larger than the coherence time. Therefore our paper was purely theoretical.

How the anti-bunching effect has it been observed now? for electrons?