# Noninteracting trapped fermions: from random matrices to stochastic growth models

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Determinantal point processes and fermions
Lille, February 6-8 2019

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in collaboration with

- David S. Dean (LOMA, Univ. of Bordeaux)
- Bertrand Lacroix-A-Chez-Toine (LPTMS, Univ. Paris Sud)
- Pierre Le Doussal (LPTENS, Ecole Normale Sup., Paris)
- Satya N. Majumdar (LPTMS, Univ. Paris-Sud)

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# Ultra-cold atoms in confining potentials

- Recent progress in the experimental manipulation of cold atoms

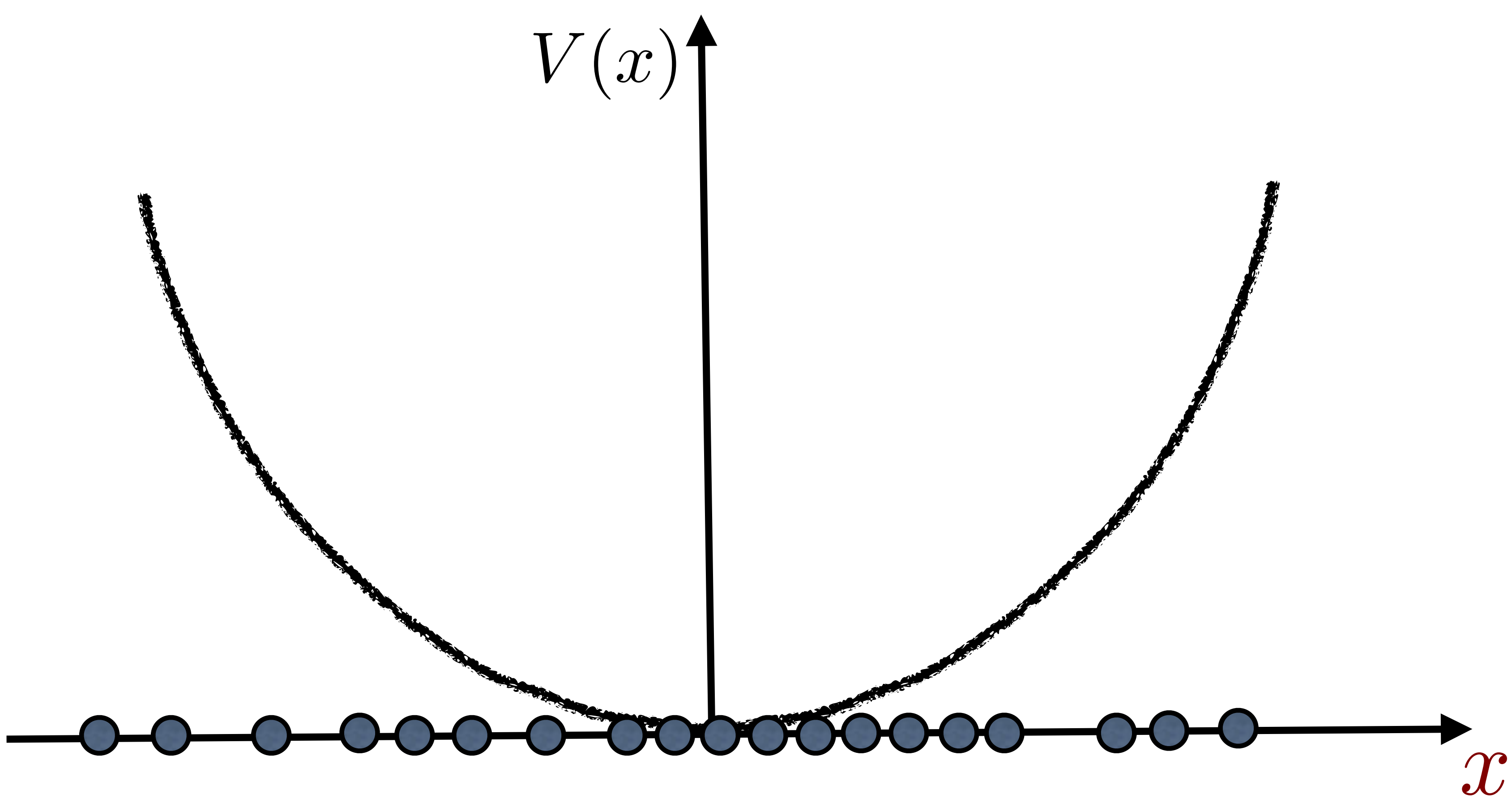
⇒ to investigate the interplay between quantum and thermal behaviors in many-body systems at low temperature

# Ultra-cold atoms in confining potentials

- Recent progress in the experimental manipulation of cold atoms

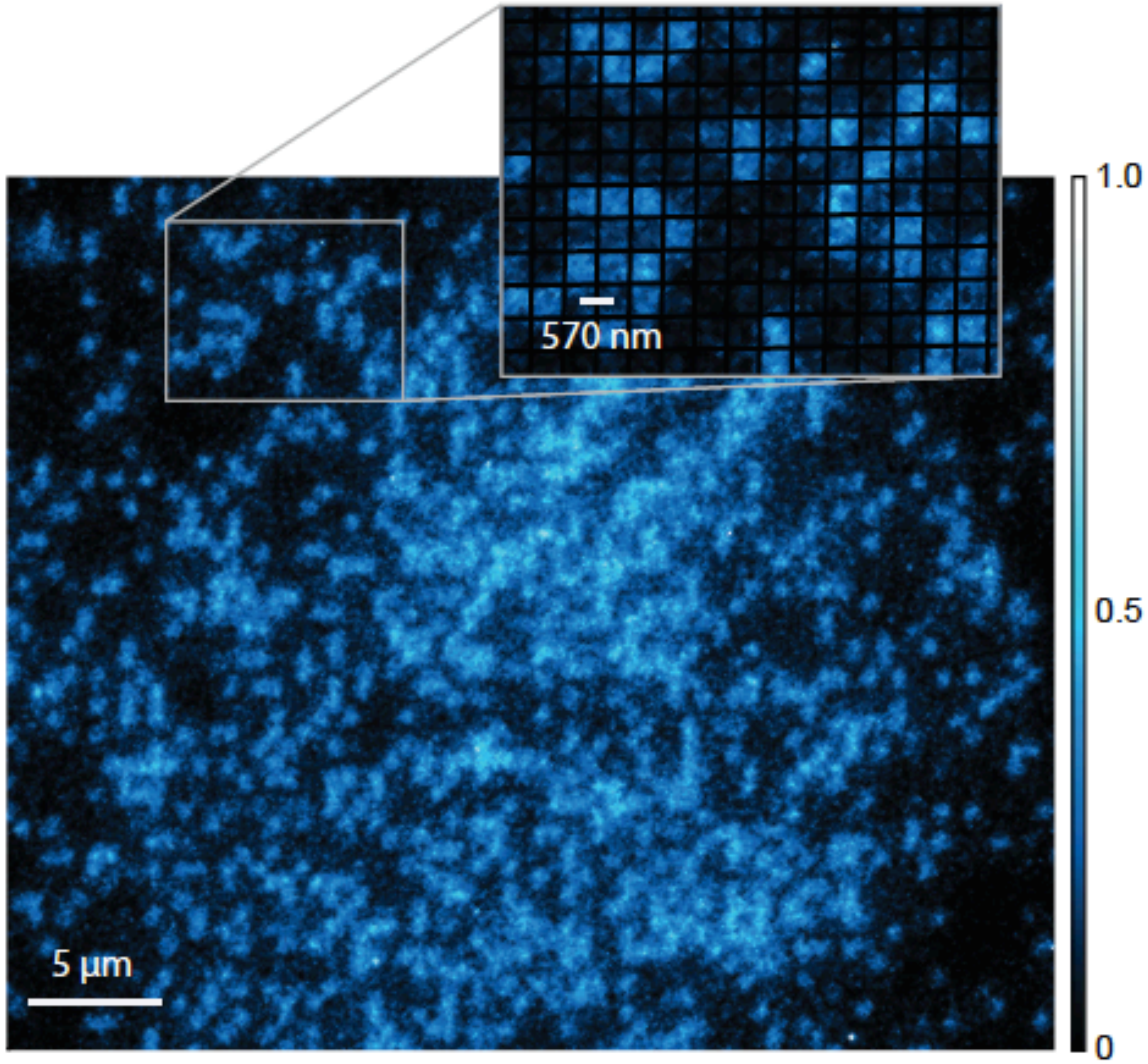
→ to investigate the interplay between quantum and thermal behaviors in many-body systems at low temperature

- A common feature of these experiments: presence of a confining potential that traps the atoms within a limited spatial region



# Quantum Fermi gas microscope

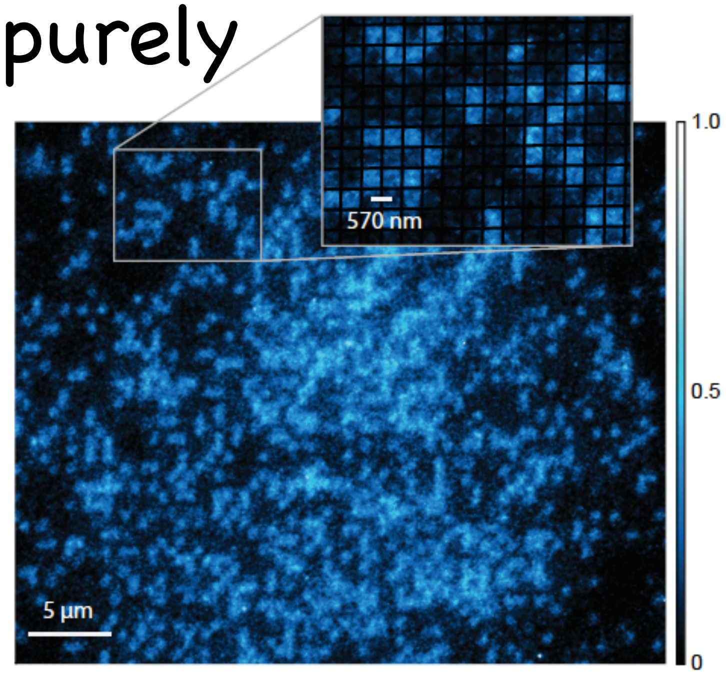
- Direct imaging of spatial fluctuations of the positions of fermions




M. Greiner et al., PRL 2015

# Tuning the interactions

- Reaching the non-interacting limit to probe purely quantum effects



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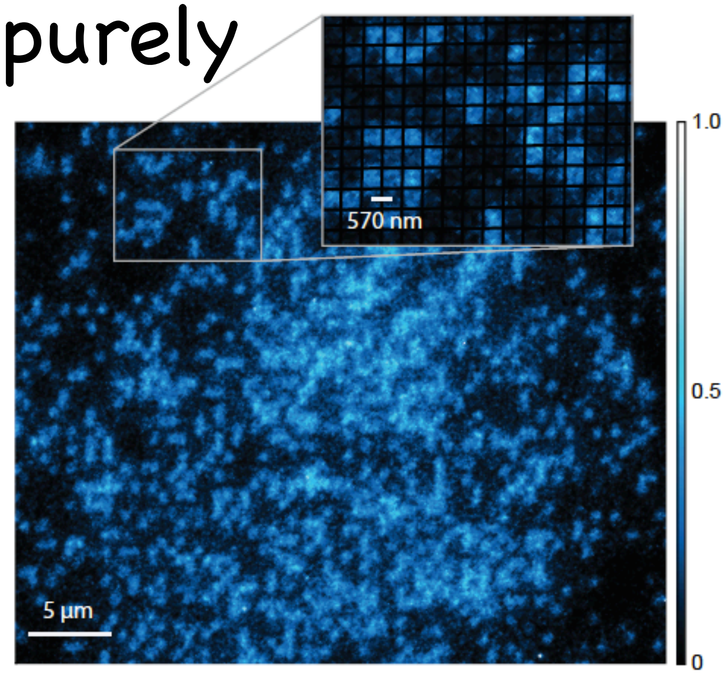
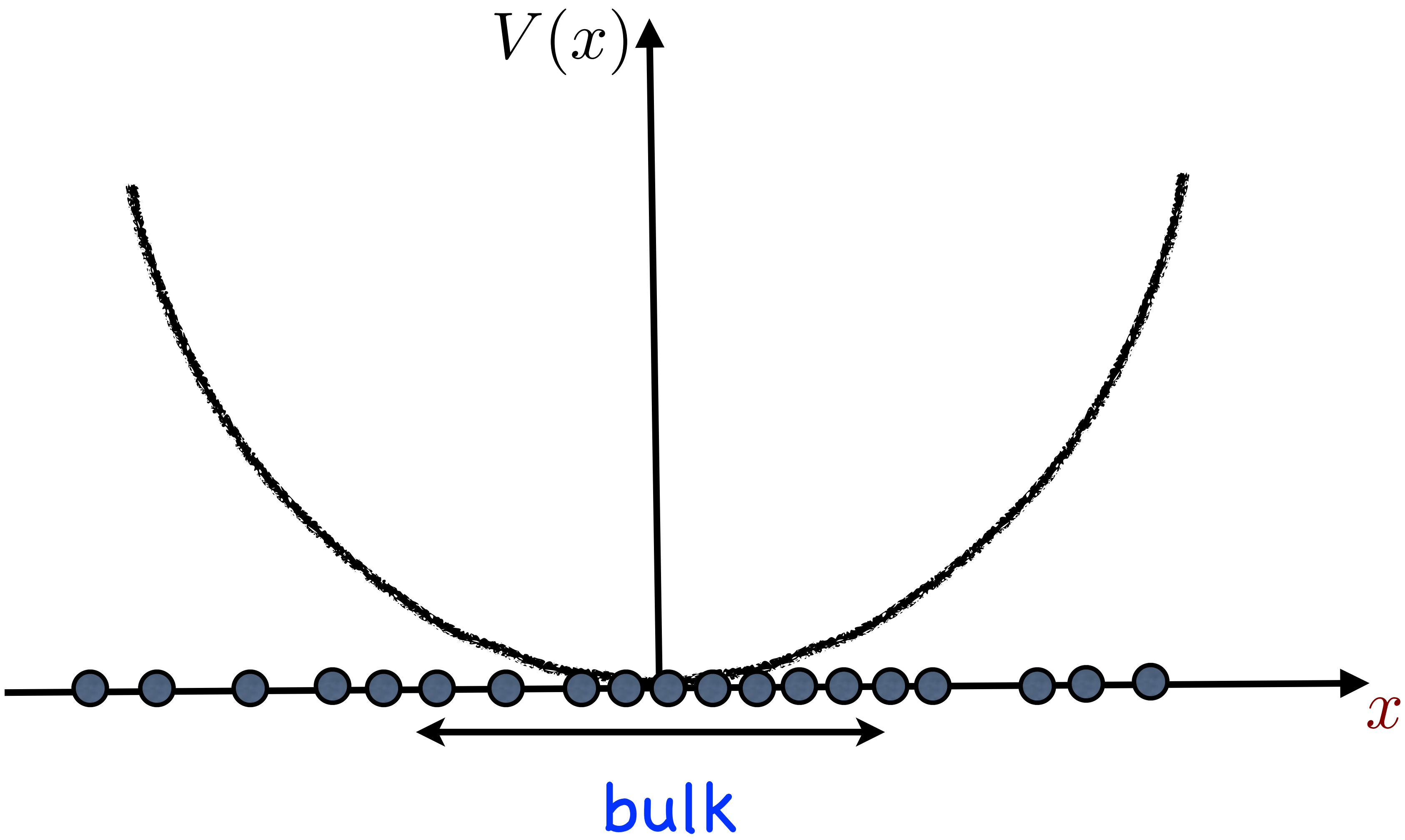



- Interesting quantum many-body effects even in the absence of interactions

Bosons: Bose-Einstein condensation

Fermions: Pauli exclusion principle → rich quantum many-body physics

# Ultra-cold atoms in confining potentials




- bulk: traditional many-body physics (translationally invariant system)

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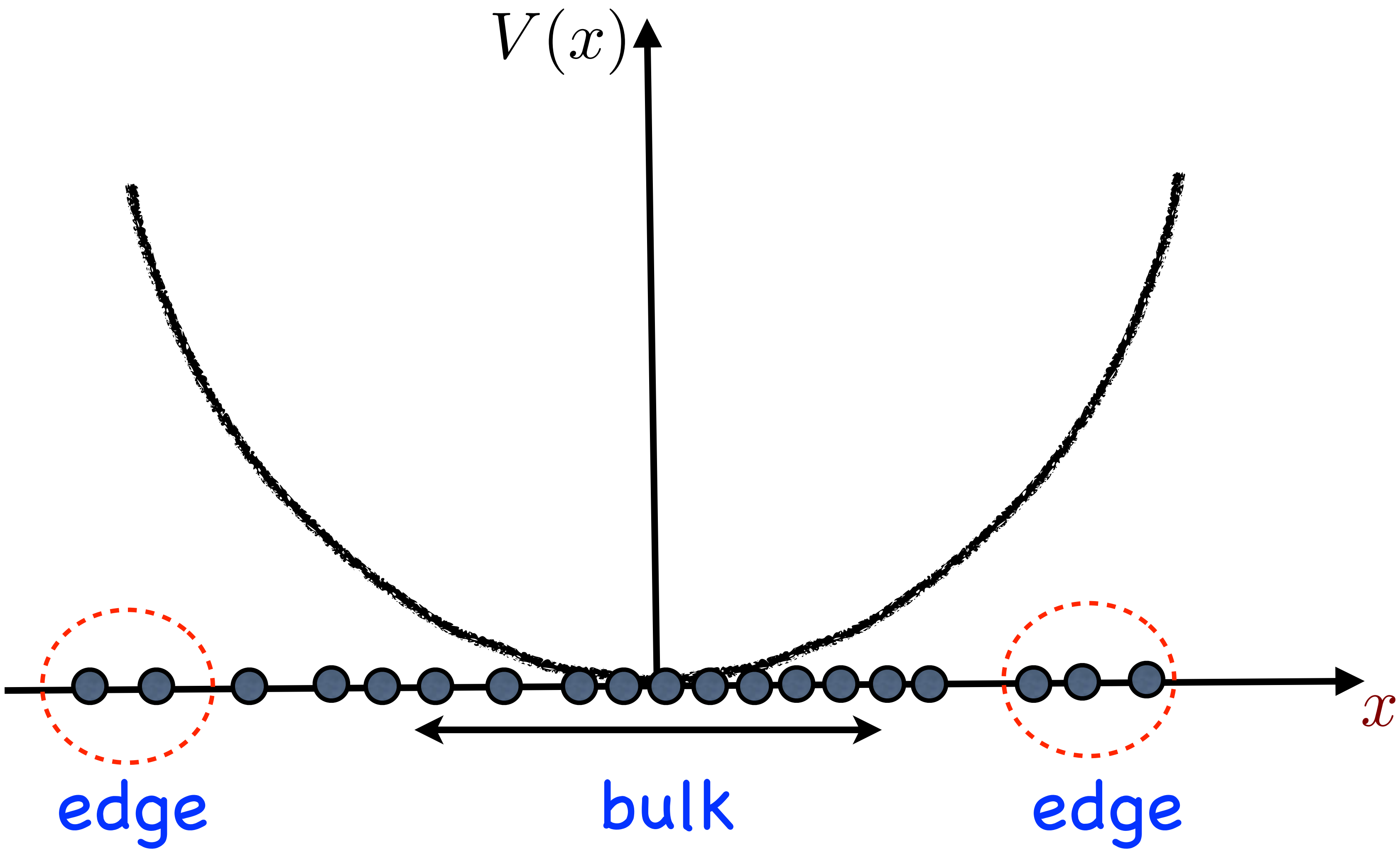


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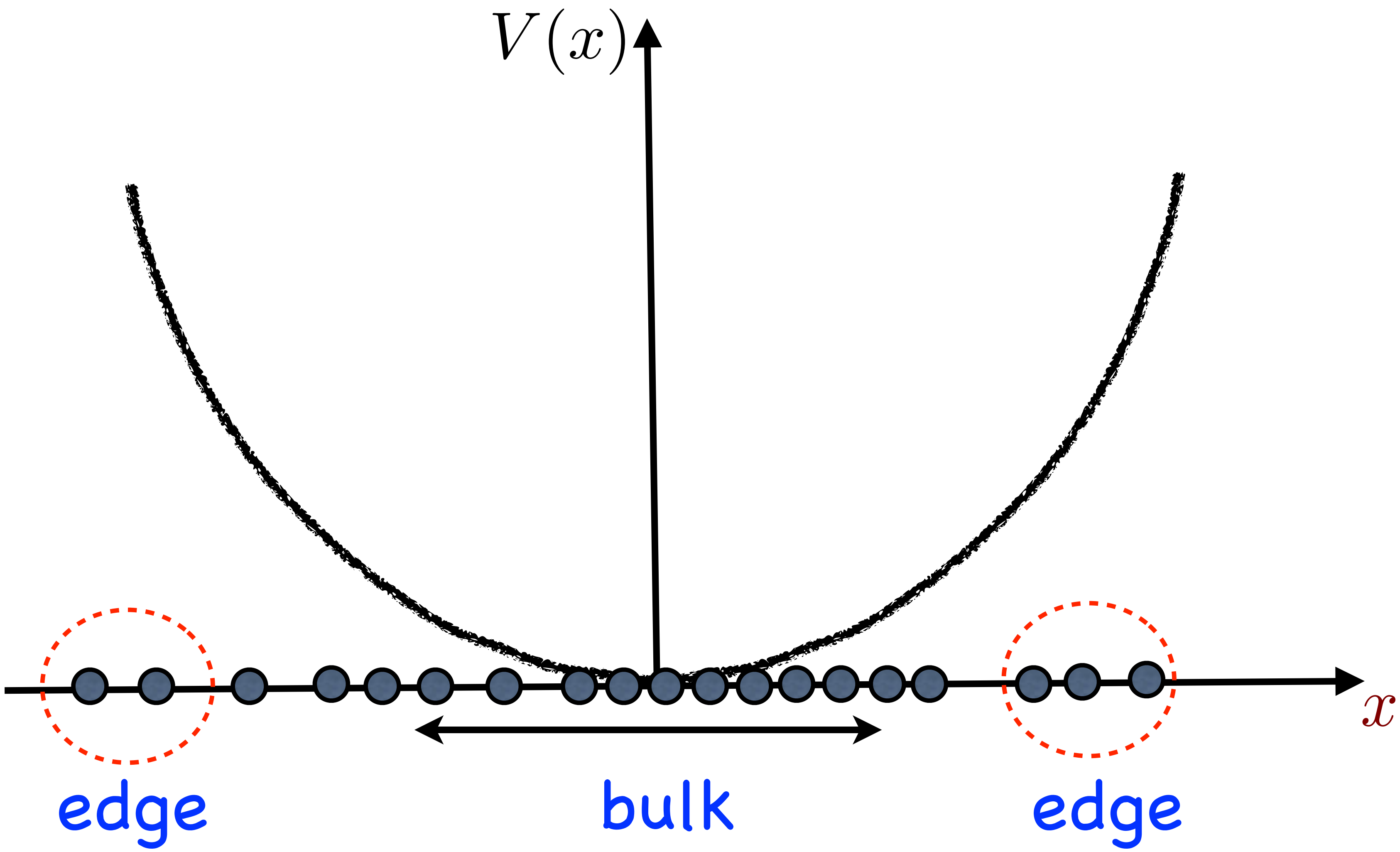


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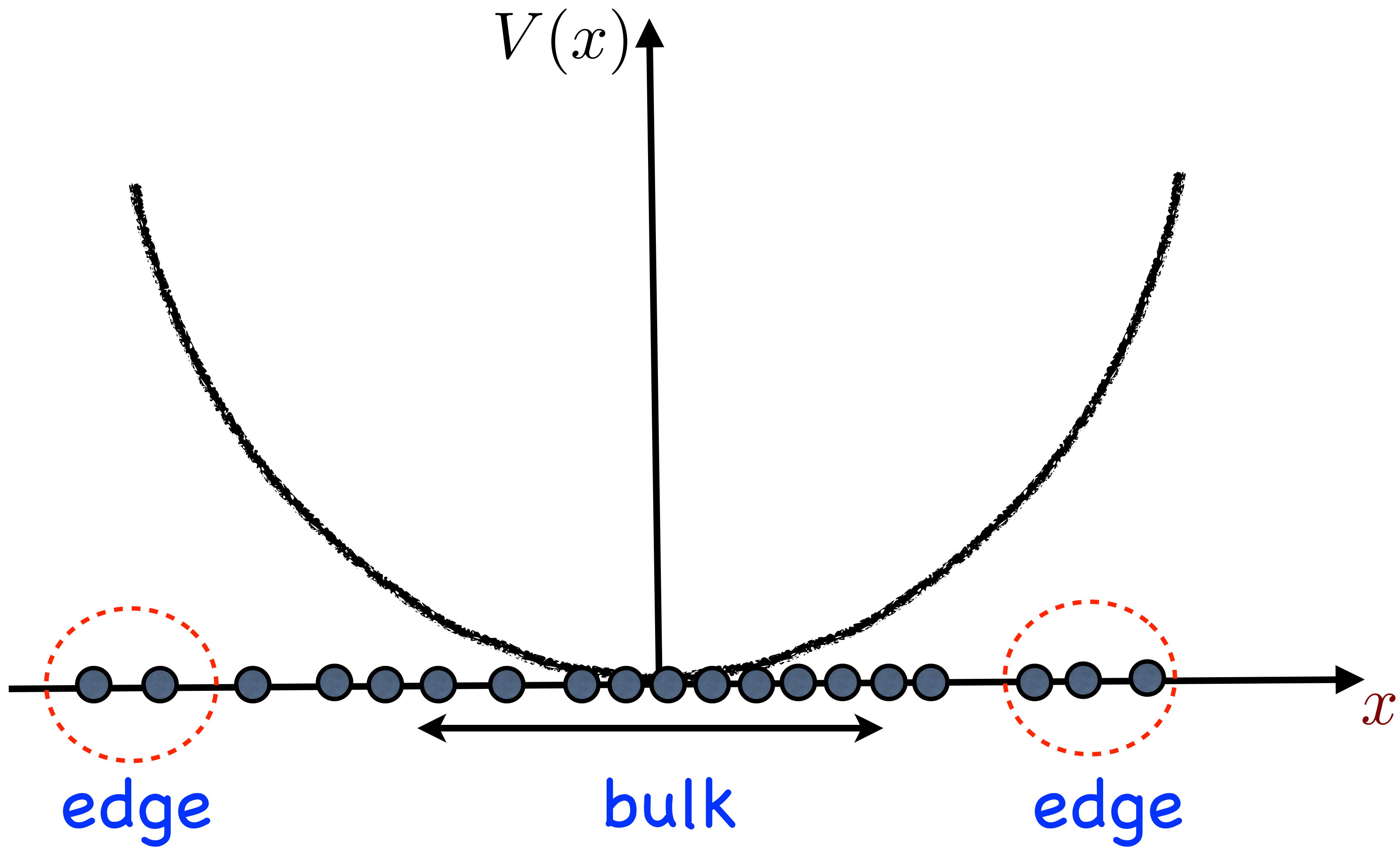


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This talk: random matrix theory is the ideal tool to study these edge properties

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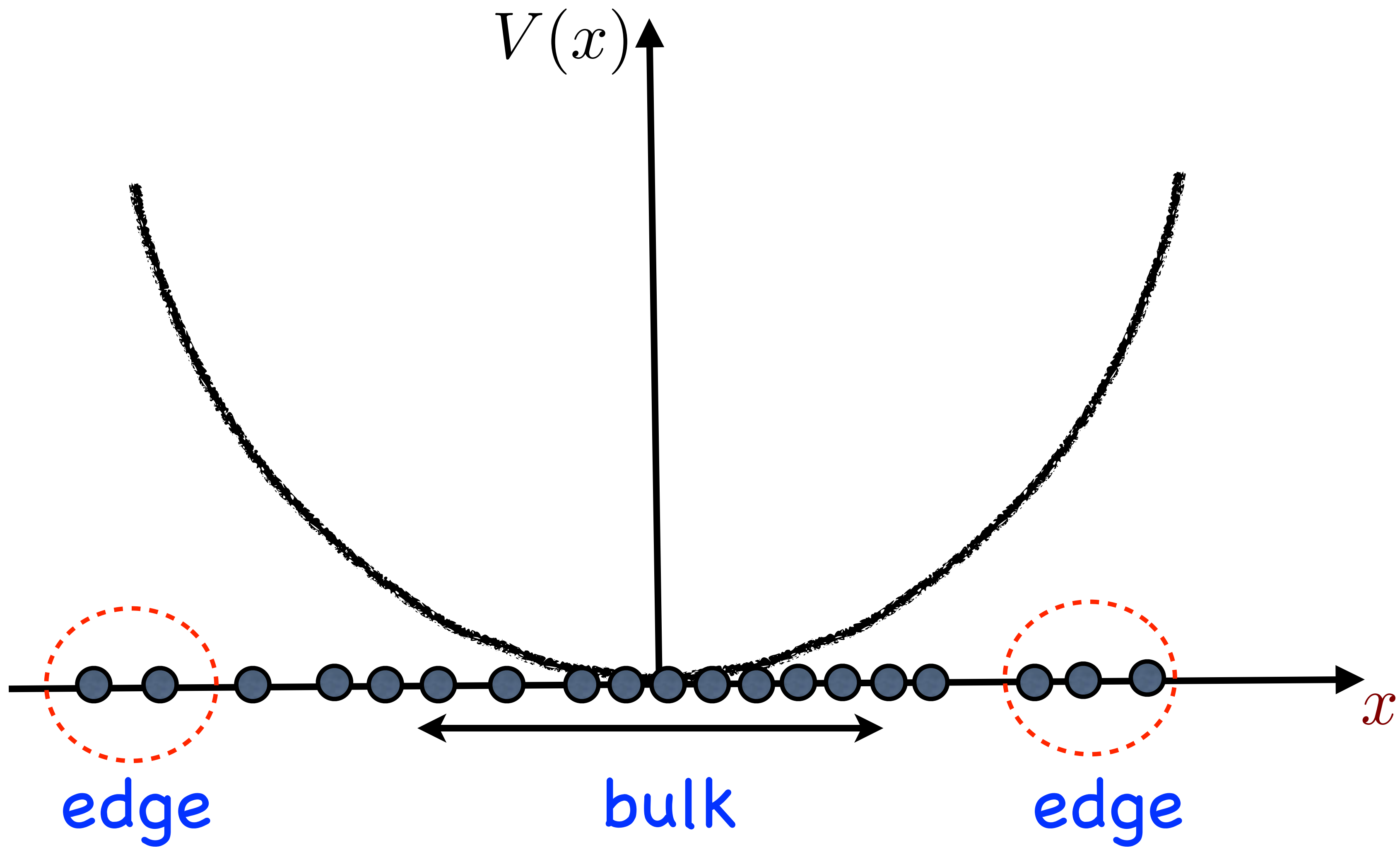


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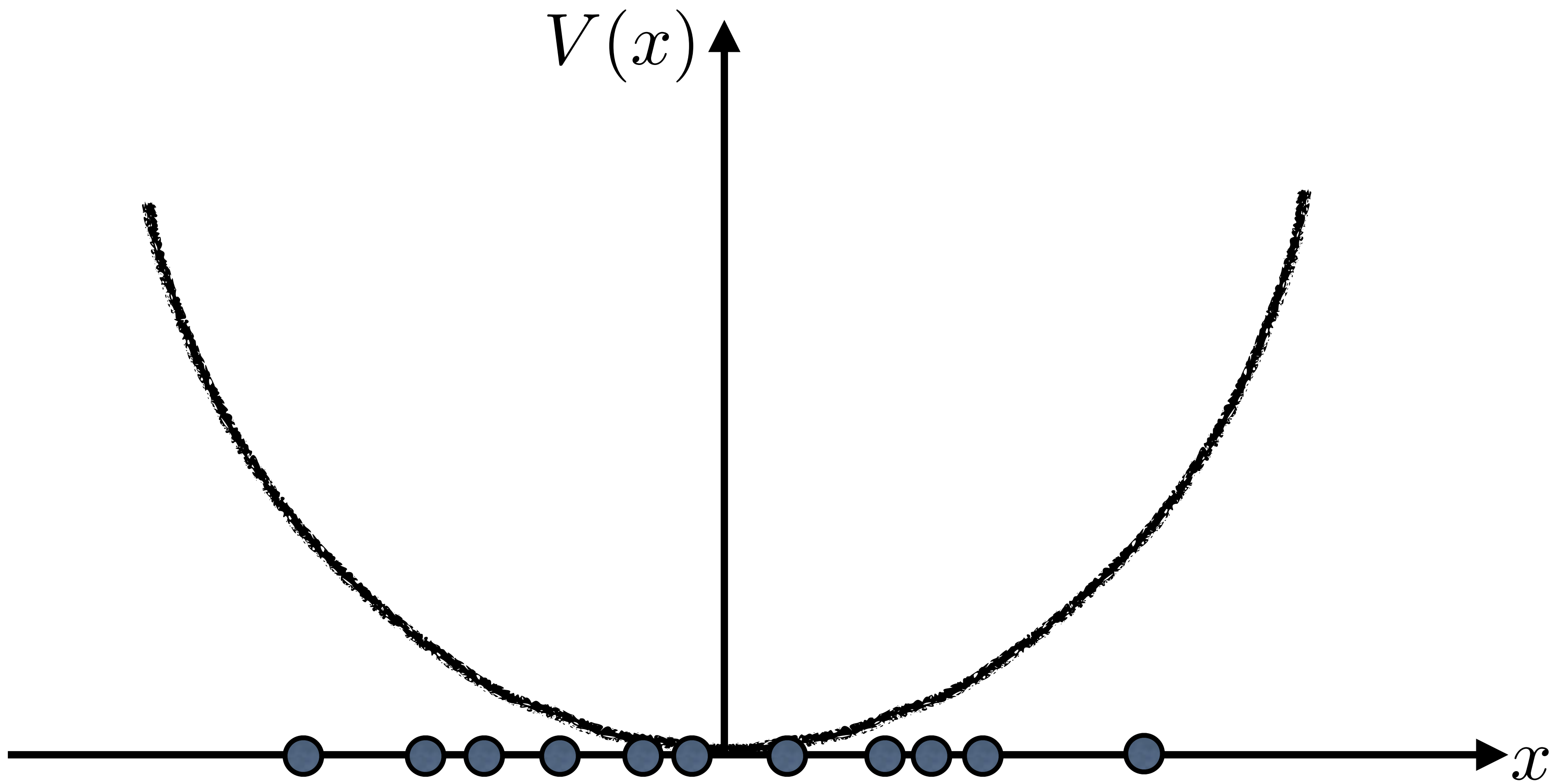
- edge: new physics induced by confinement ⟹ universal edge properties

This talk: random matrix theory is the ideal tool to study these edge properties

# Spinless free fermions in a 1d harmonic potential




$$V(x) = \frac{1}{2} m \omega^2 x^2$$

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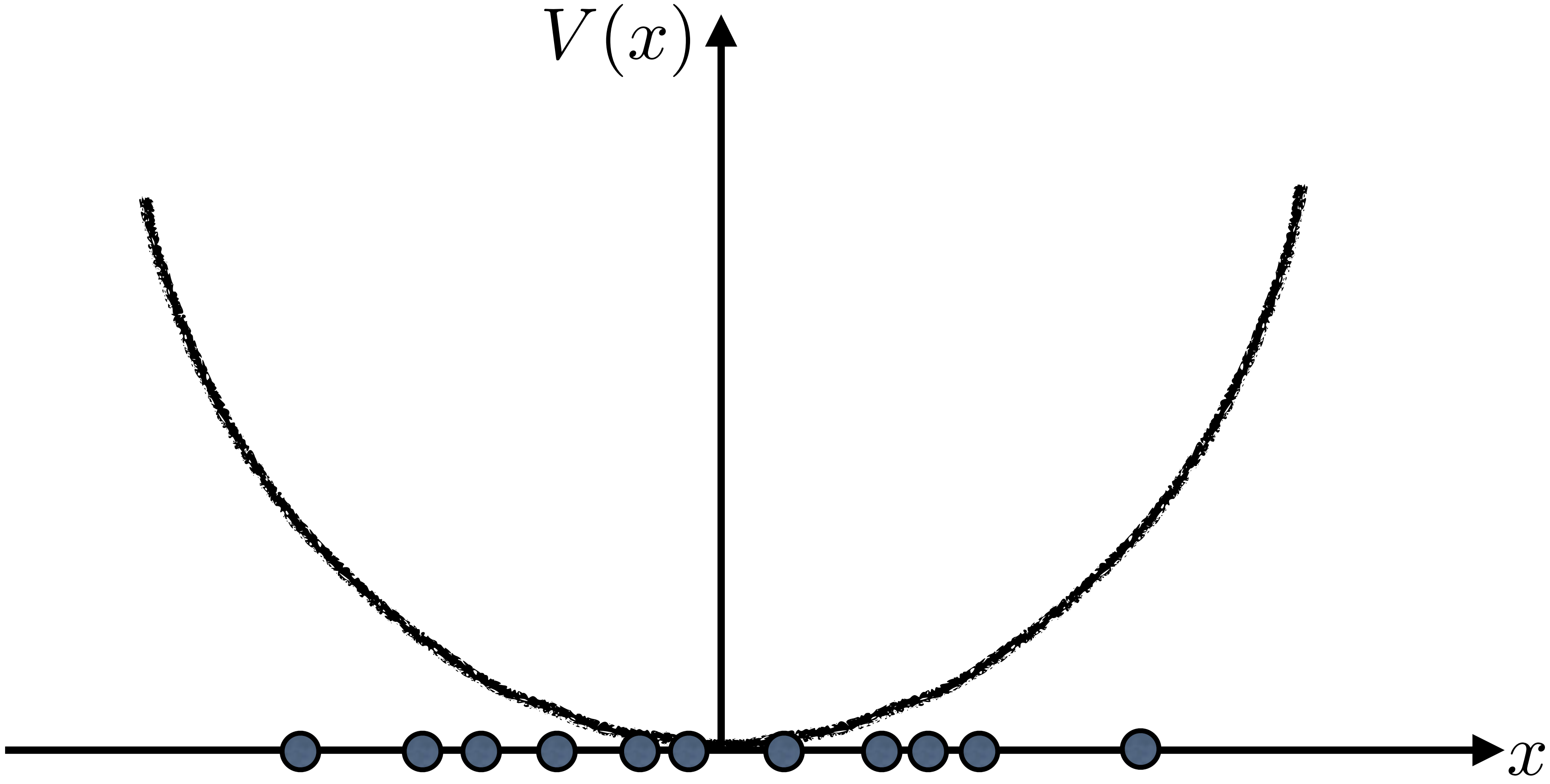


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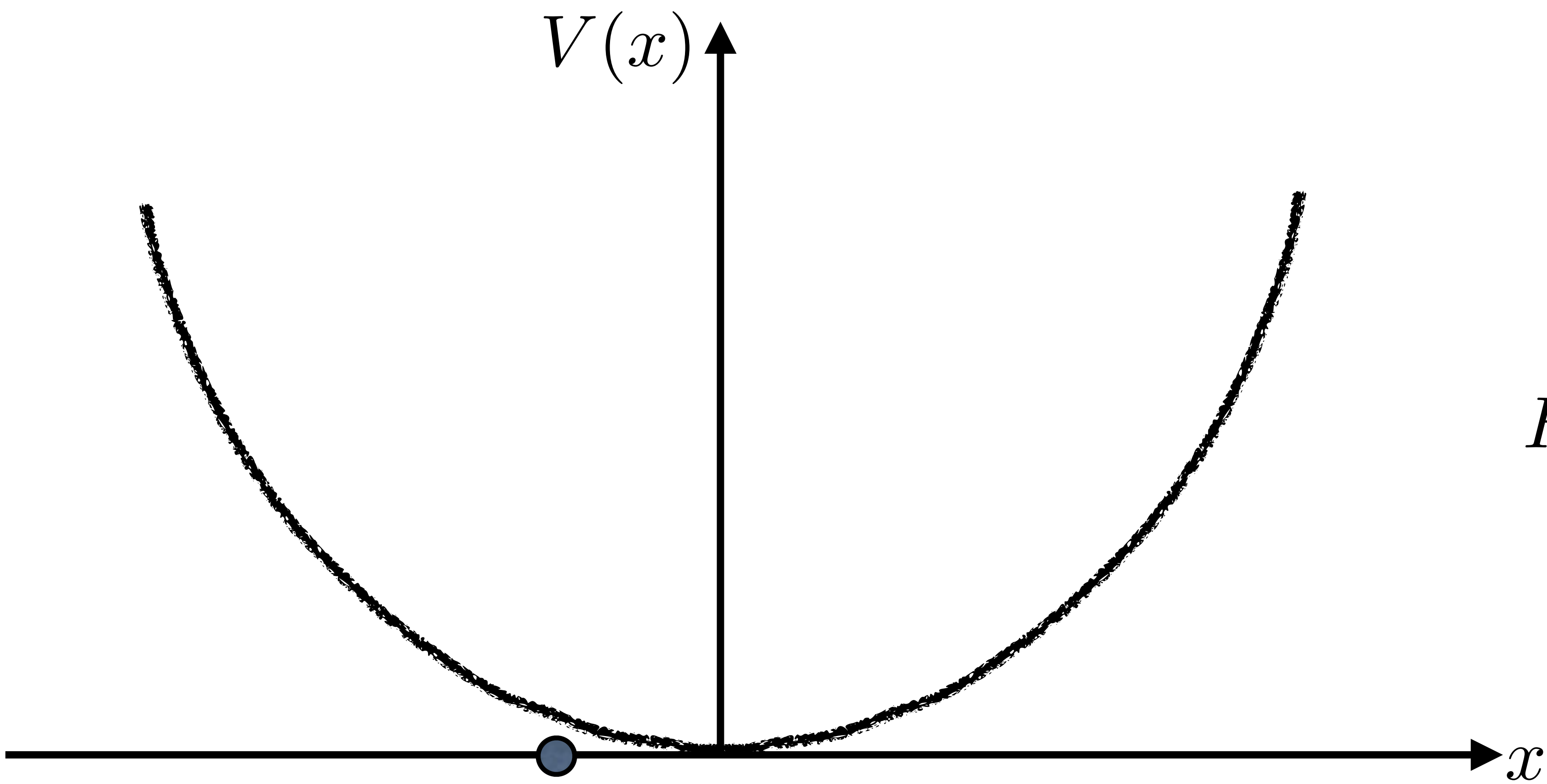
At zero temperature: connection between spinless free fermions in a harmonic trap and Random Matrix Theory (GUE)

# Connection between free fermions at T=0 and RMT

- A single quantum particle in a harmonic potential

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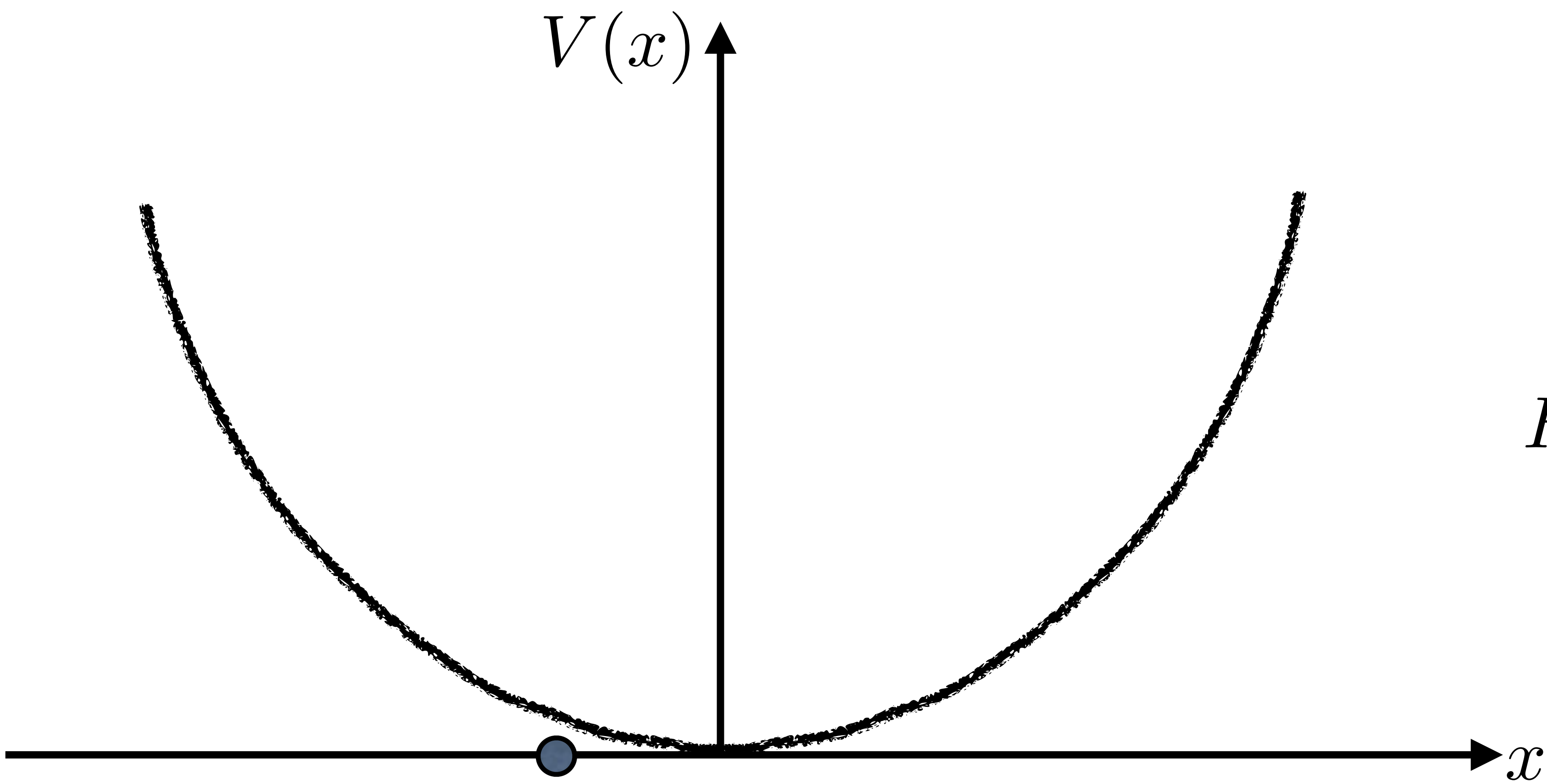


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- Single particle eigenfunctions

$$\hat{H}\,\varphi_E(x) = E\,\varphi_E(x)$$

with $\varphi_E(x \to \pm\infty) = 0$

$$\varphi_k(x) = \left[\frac{\alpha}{\sqrt{\pi}2^k k!}\right]^{1/2} e^{-\frac{\alpha^2 x^2}{2}} H_k(\alpha x)$$

$$\epsilon_k = \hbar\omega(k+1/2) \quad , \quad \alpha = \sqrt{m\omega/\hbar}$$

$$k \in \mathbb{N}$$

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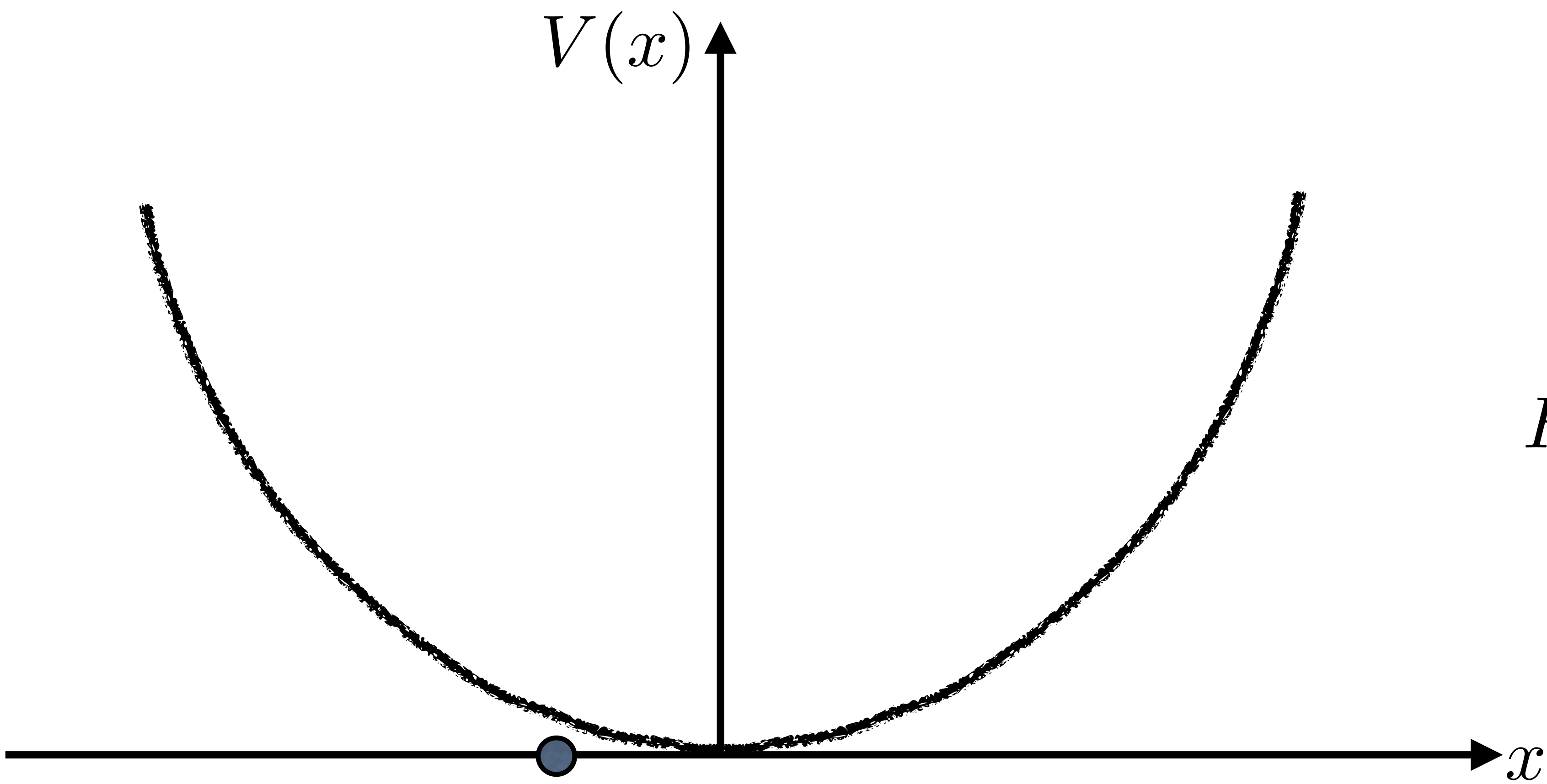


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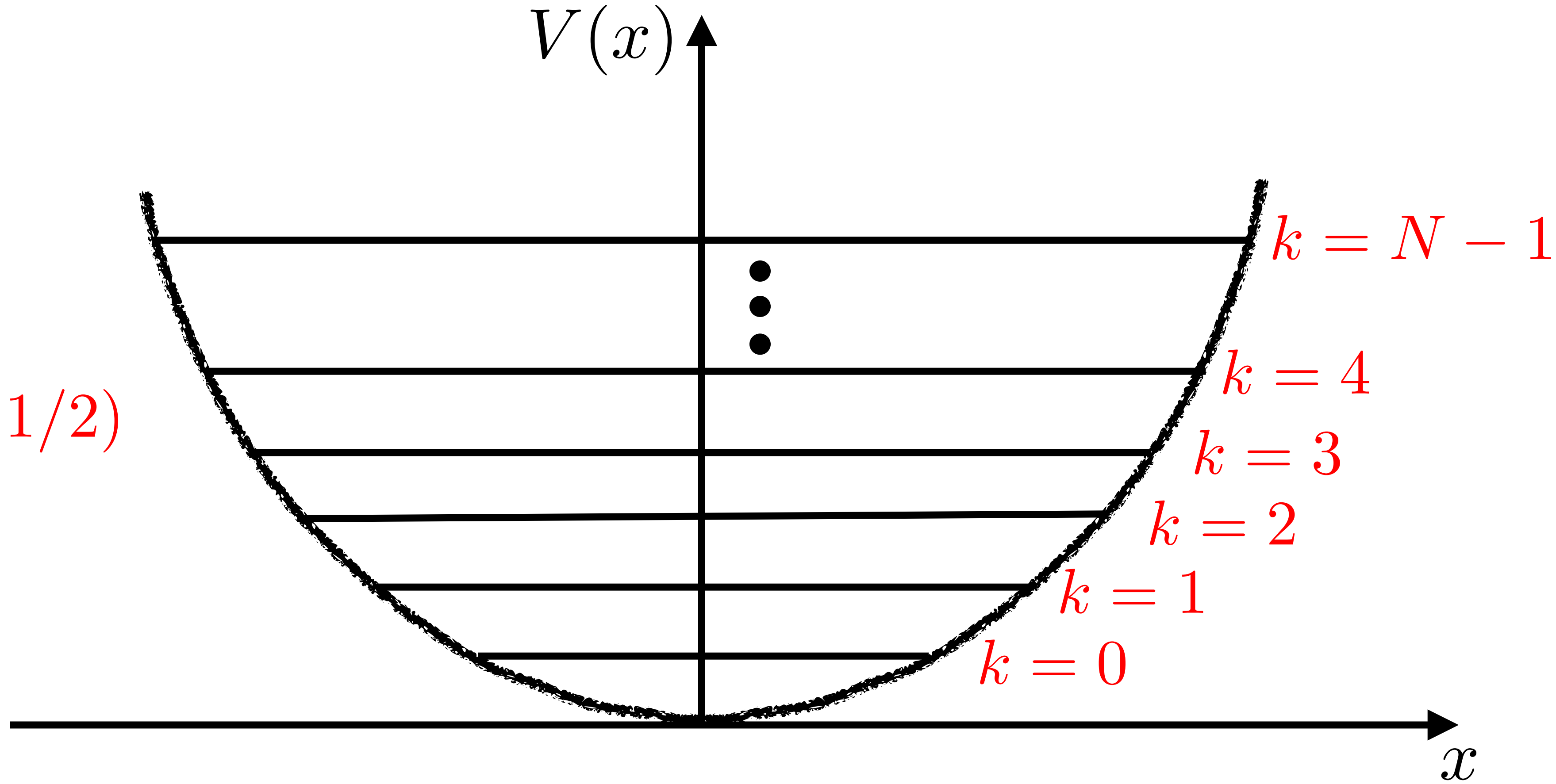


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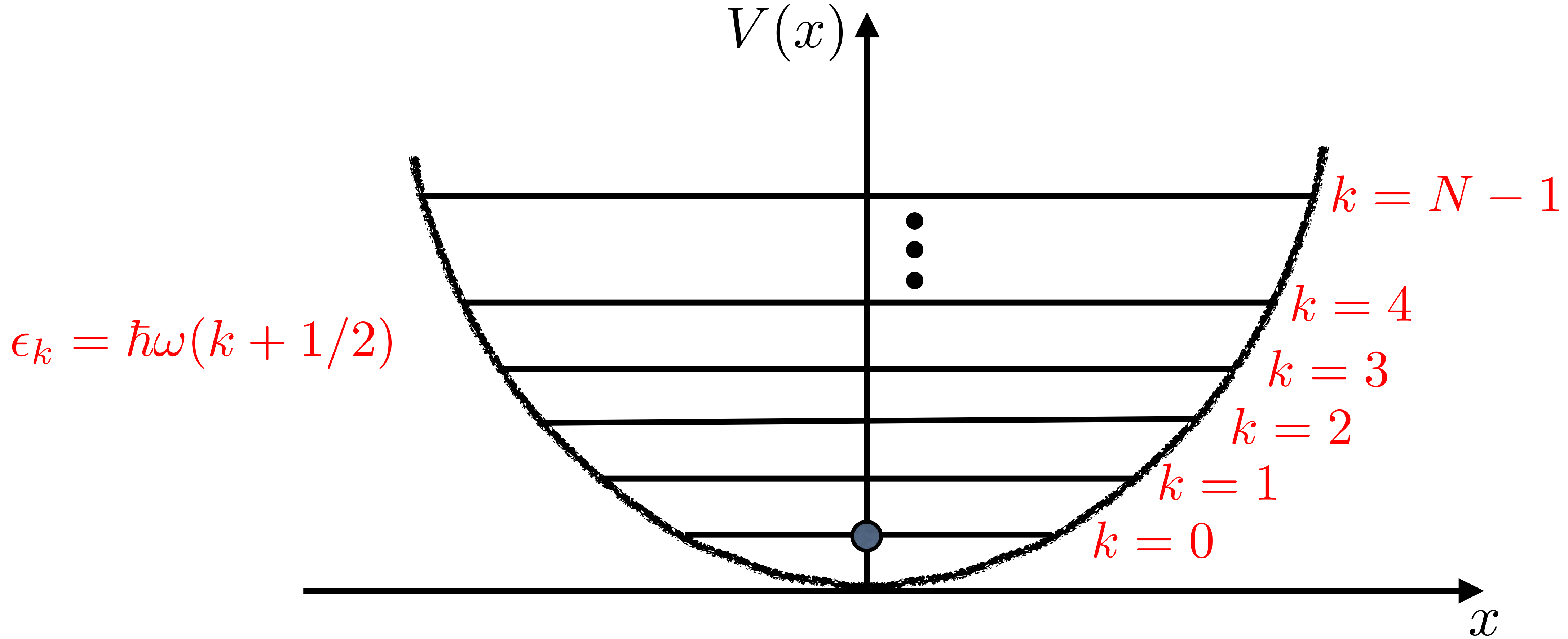

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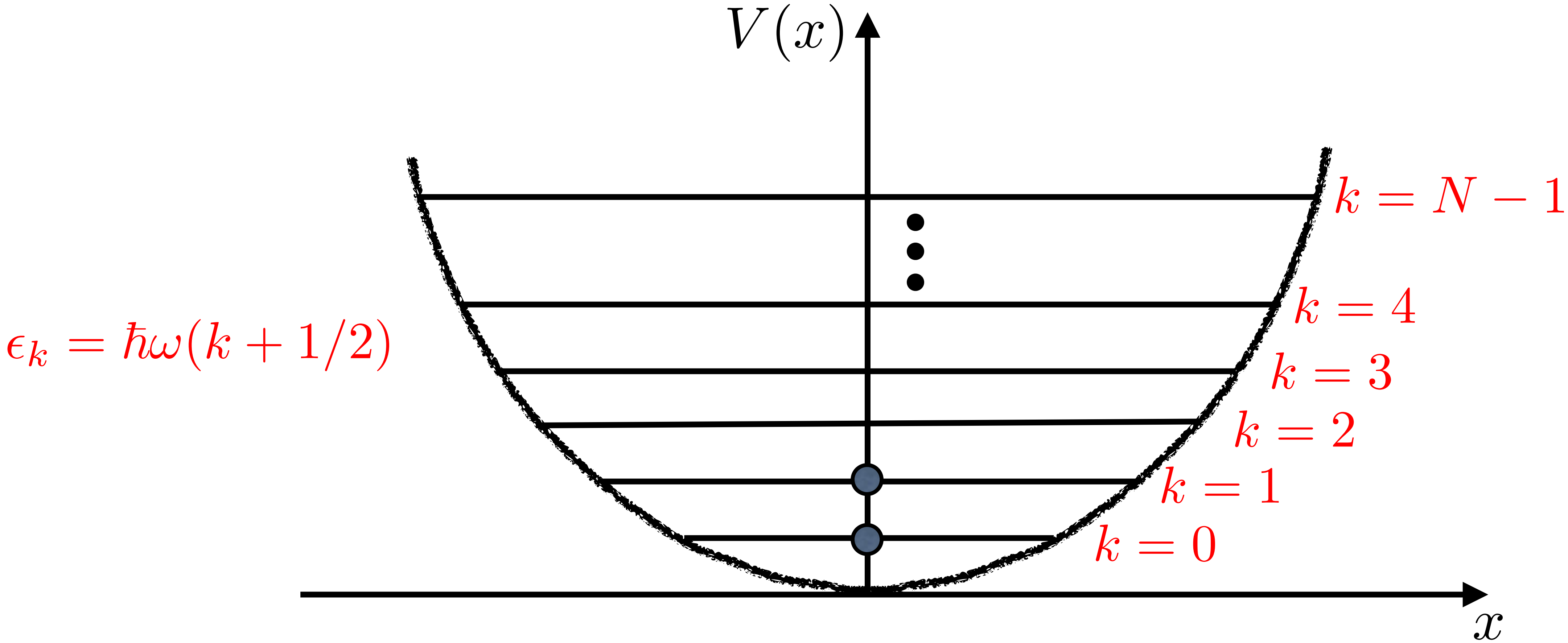

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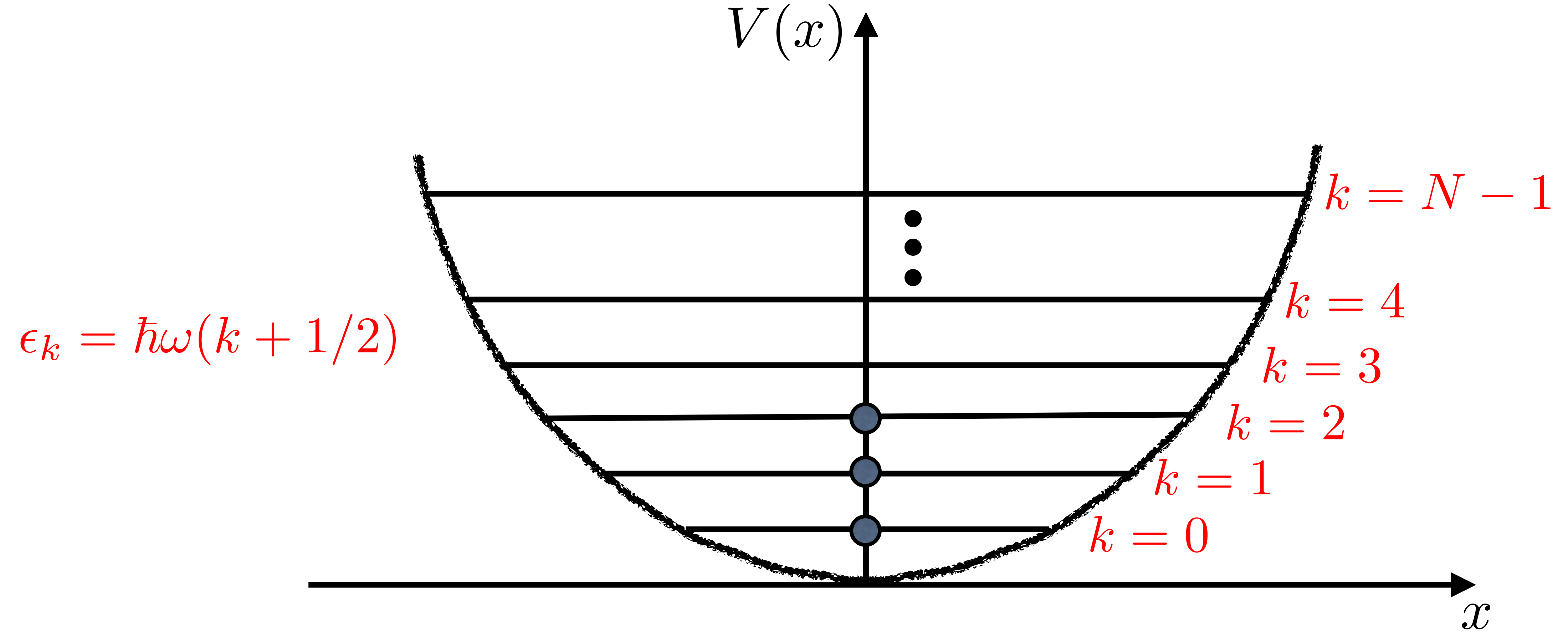

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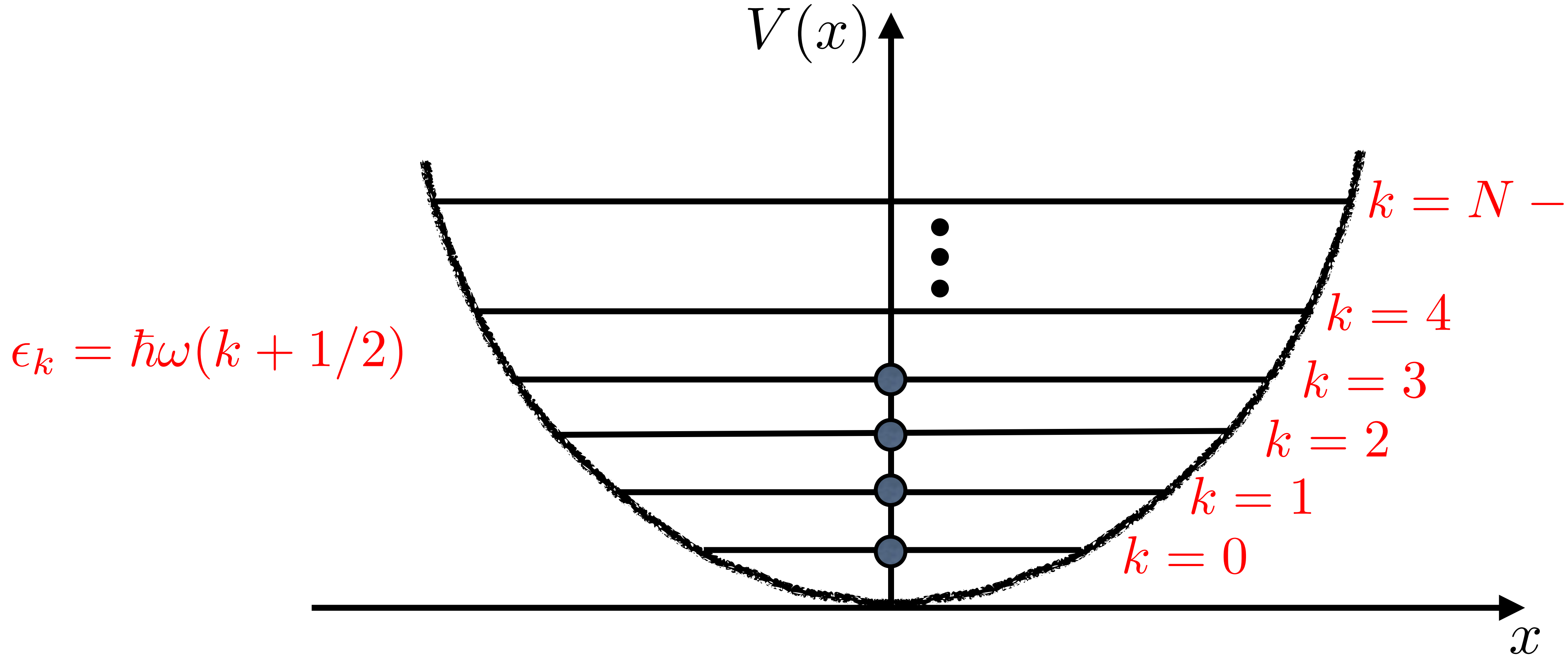

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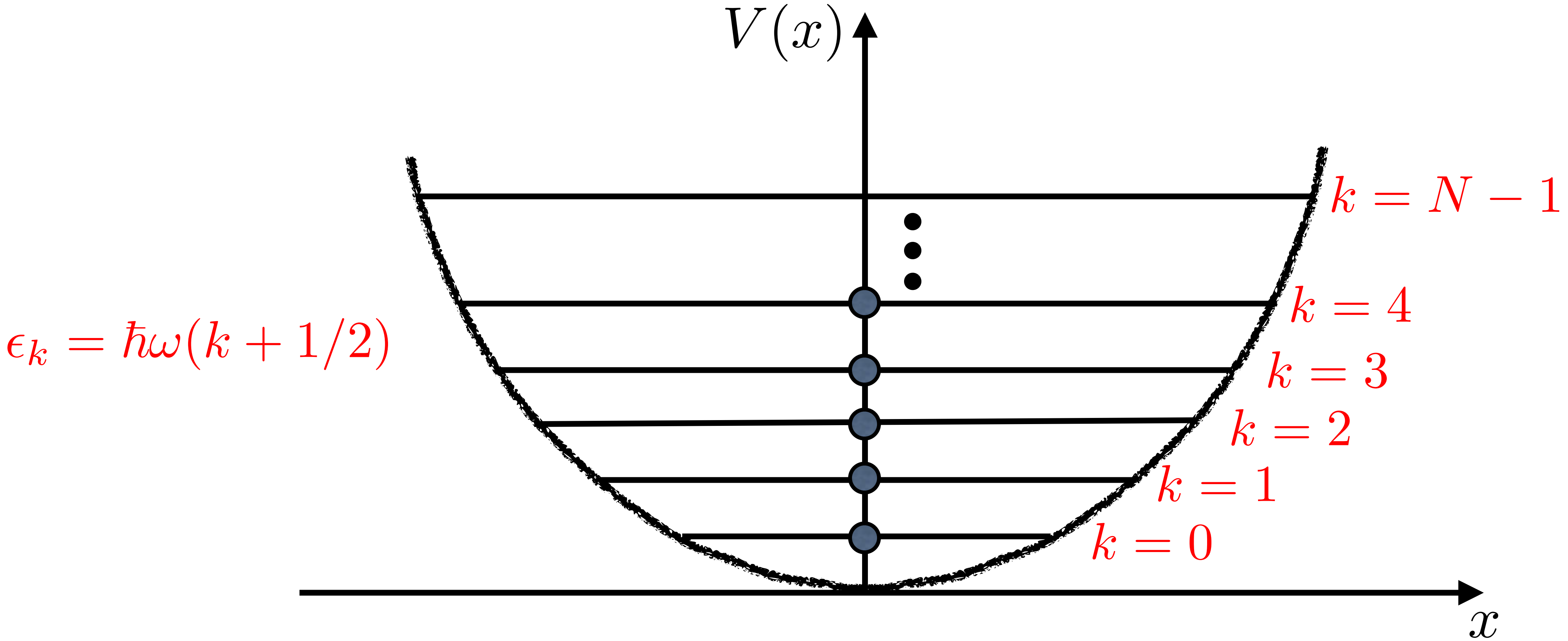

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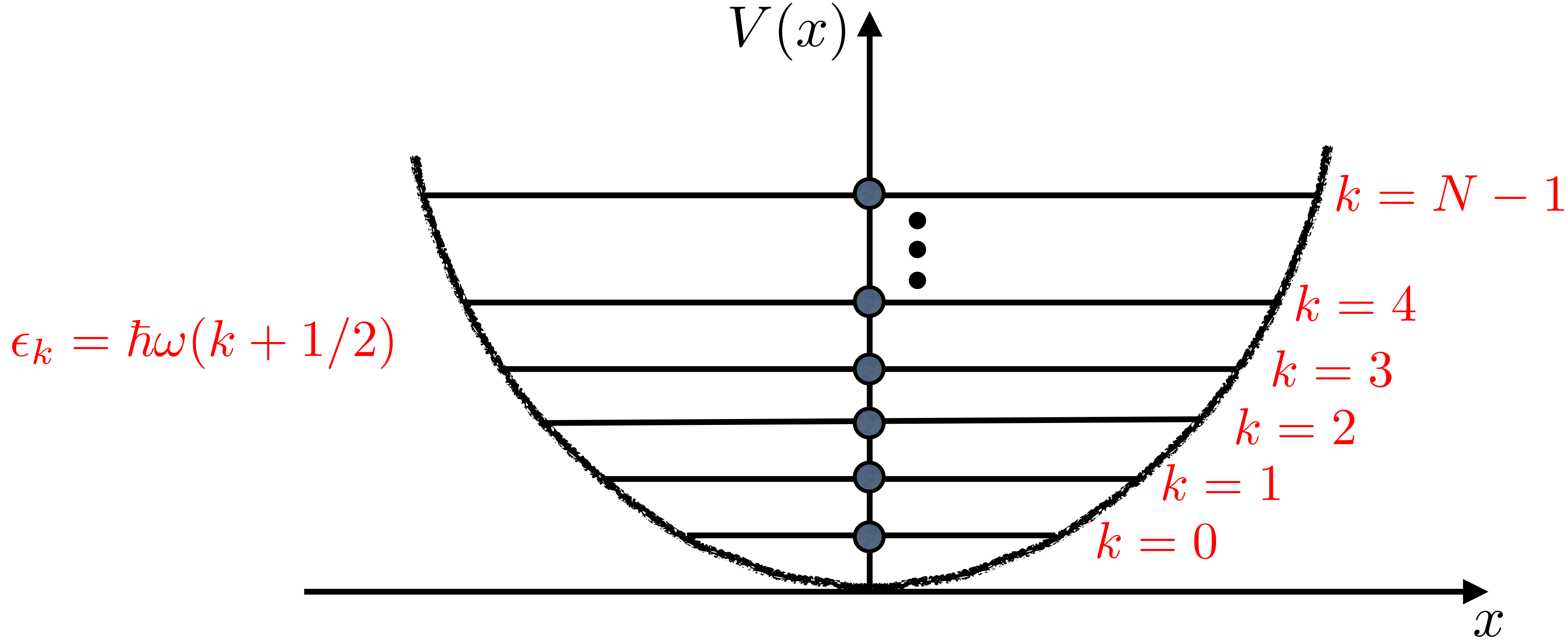

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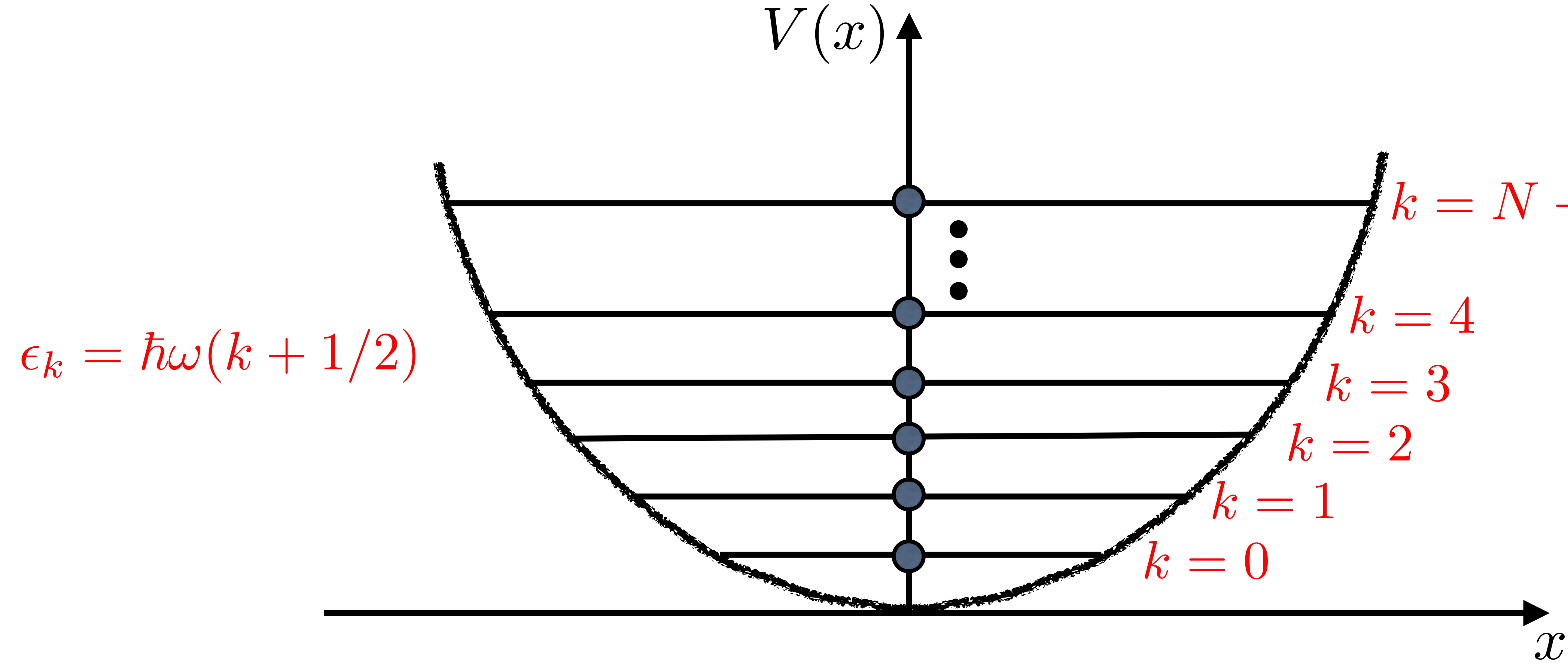


- The $N$-particle wave function is given by a $N \times N$ Slater determinant

$$\Psi_0(x_1, x_2, \cdots, x_N) = \frac{1}{\sqrt{N!}} \det[\varphi_i(x_j)] \qquad \begin{array}{l} 0 \leq i \leq N-1 \\ 1 \leq j \leq N \end{array}$$

$$\varphi_k(x) = \left[\frac{\alpha}{\sqrt{\pi} 2^k k!}\right]^{1/2} e^{-\frac{\alpha^2 x^2}{2}} H_k(\alpha x)$$

# Connection between free fermions at T=0 and RMT

- Ground-state wave function

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↑ Hermite polynomial of degree $i$

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- Probability density function (PDF) of the positions $x_i's$

$$|\Psi_0(x_1, \cdots, x_N)|^2 = \frac{1}{z_N(\alpha)} \prod_{i<j} (x_i - x_j)^2 e^{-\alpha^2 \sum_{i=1}^N x_i^2}$$

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- Squared many-body wave function (T=0 quantum probability) for fermions

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- Let $J$ be a $N \times N$ random Hermitian matrix with Gaussian (complex) entries. The PDF of the (real) eigenvalues $\lambda_i' s$ is given by

$$P_{\text{joint}}(\lambda_1,\cdots,\lambda_N) = \frac{1}{Z_N} \prod_{i<j} (\lambda_i - \lambda_j)^2 e^{-\sum_{i=1}^N \lambda_i^2}$$

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→ The positions of the free fermions behave statistically like the eigenvalues of GUE random matrices

# What about other (unitary) matrix models ?

- Laguerre Unitary Ensemble can be realized with a singular potential

$$V(x) = \frac{\alpha(\alpha-1)}{x^2} + \beta x^2 \; , \; x > 0$$

Nadal, Majumdar, PRE '09

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$$V(x) = \frac{\alpha(\alpha - 1)}{x^2} + \beta x^2 \; , \; x > 0$$

Nadal, Majumdar, PRE '09

- Jacobi Unitary Ensemble can be realized with a box potential

$$V(x) = \begin{cases} 0 \; , & -1 \leq x \leq +1 \\ +\infty \; , & |x| > 1 \end{cases}$$

Lacroix-A-Chez-Toine, Le Doussal, Majumdar, G. S., EPL '17

Dean, Le Doussal, Majumdar, G. S., arXiv:1810.12583

# Properties of fermions in a 1d harmonic trap at T=0

- Probability density function (PDF) of the positions of the fermions $x_i's$

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⇒ The spatial properties of free fermions in a harmonic trap at T=0 can directly be obtained from the known results in RMT

Eisler '13/Marino, Majumdar, G. S., Vivo '14/Calabrese, Le Doussal, Majumdar '15

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- Average density of free fermions: Wigner semi-circle law

$$\rho_N(x, T=0) = \frac{1}{N} \sum_{i=1}^N \langle \delta(x - x_i) \rangle$$

for $N \gg 1$ $\quad \rho_N(x, T=0) \approx \frac{\alpha}{\sqrt{N}} f_W\left(\frac{\alpha\, x}{\sqrt{N}}\right)$ , $f_W(z) = \frac{1}{\pi}\sqrt{2 - z^2}$

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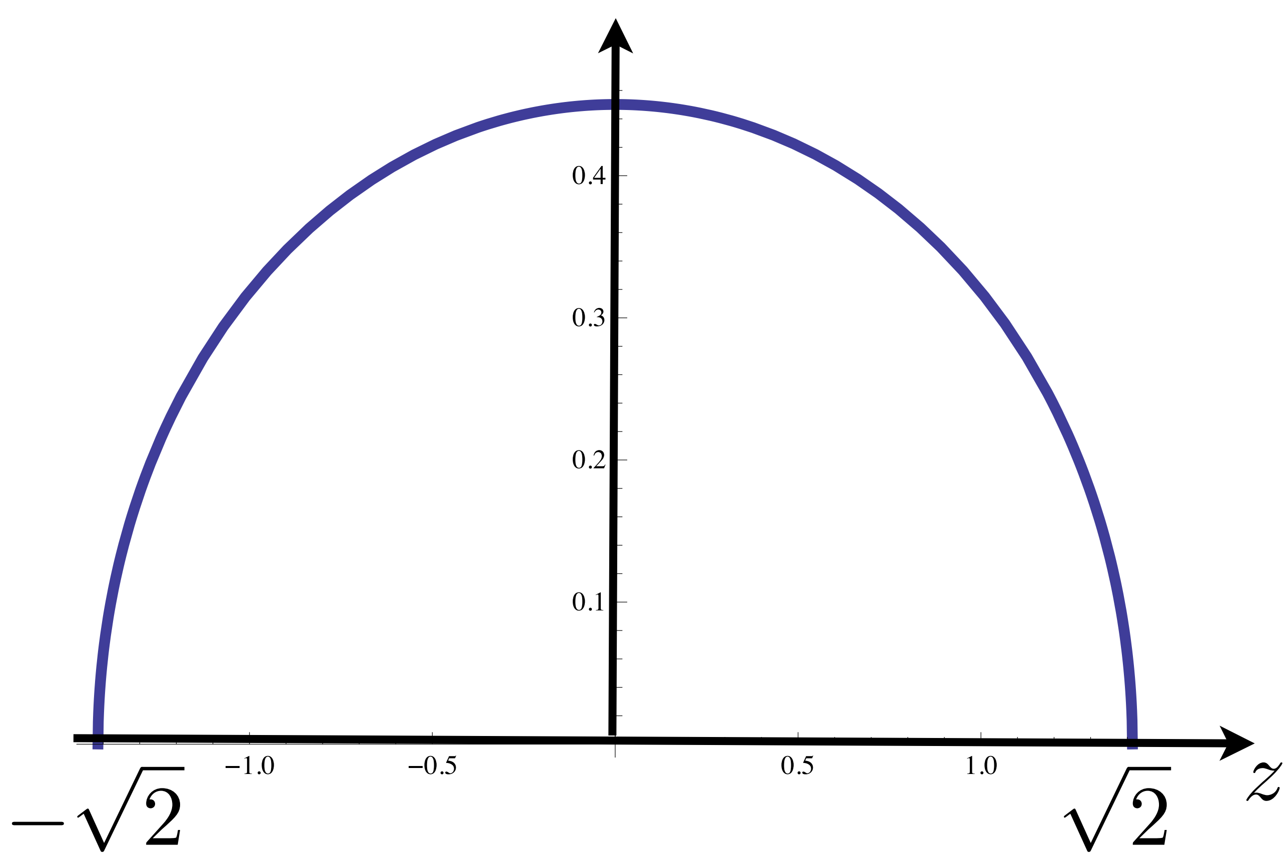
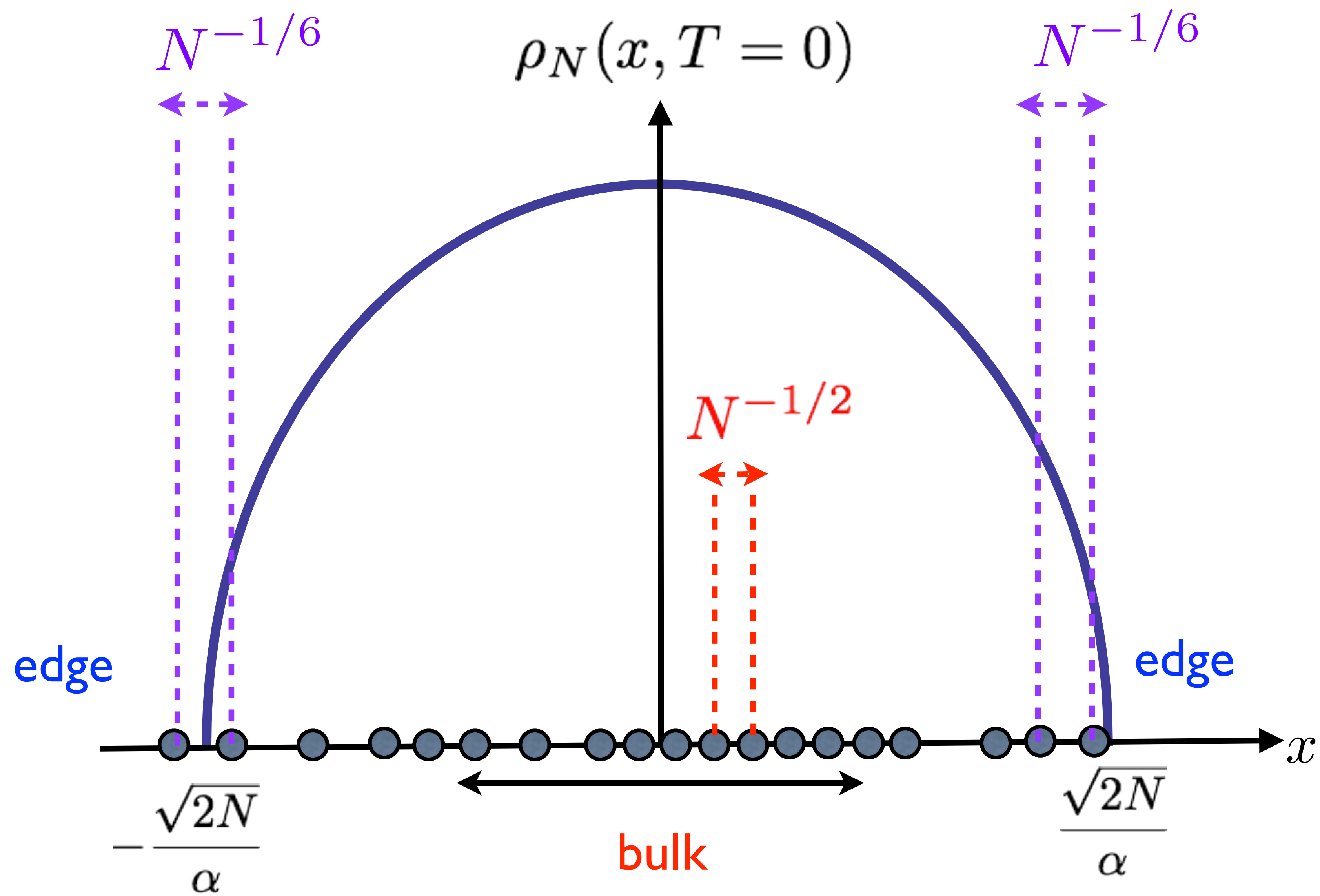
See also Local Density (or Thomas-Fermi) Approx. in the literature on fermions



# Average density of fermions at T=0: two scales




- Average density: $\rho_N(x, T=0) \sim \frac{\alpha^2}{2N}\sqrt{\frac{2N}{\alpha^2} - x^2}\ , \ \ r_{\rm edge} = \sqrt{2N}/\alpha$

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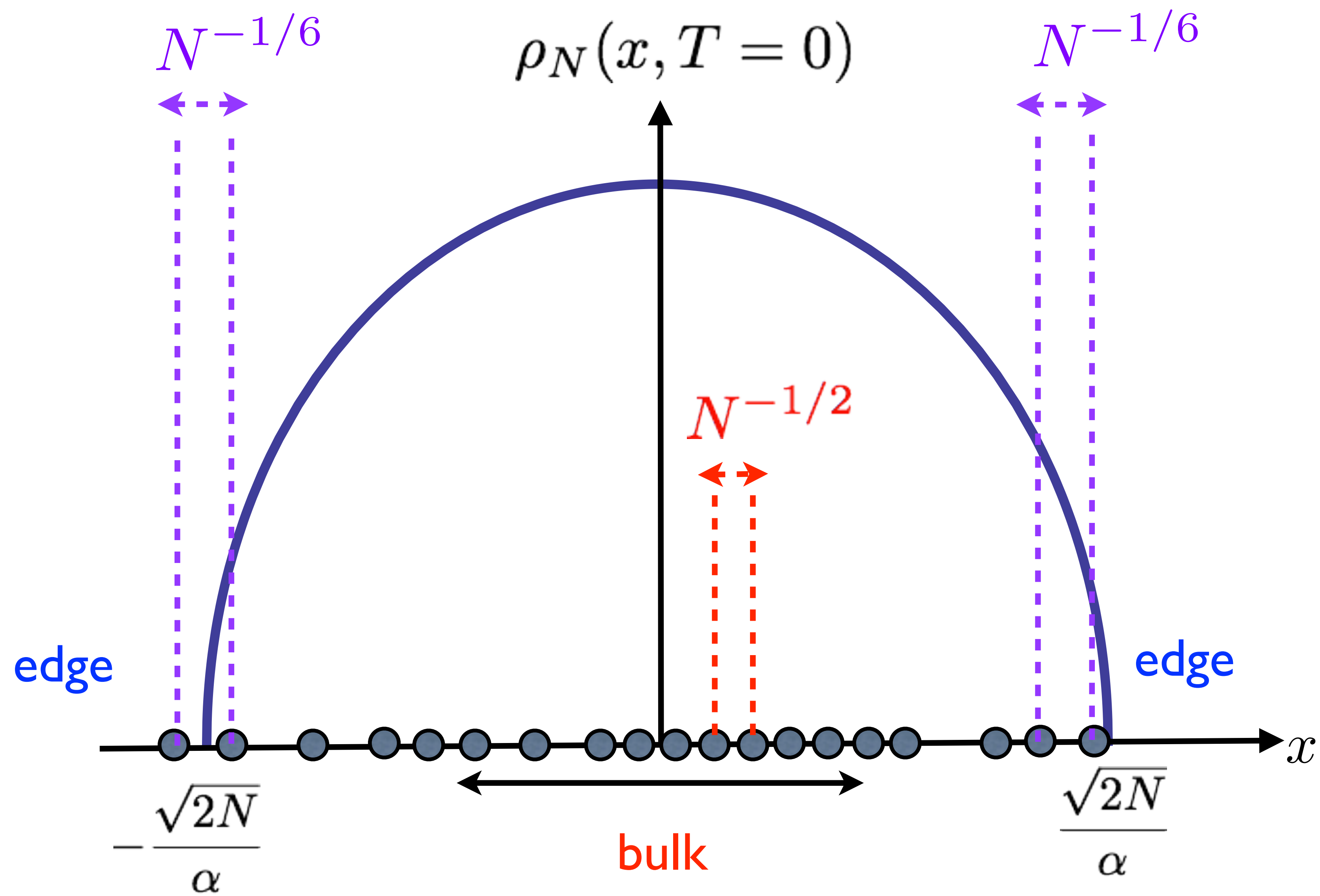


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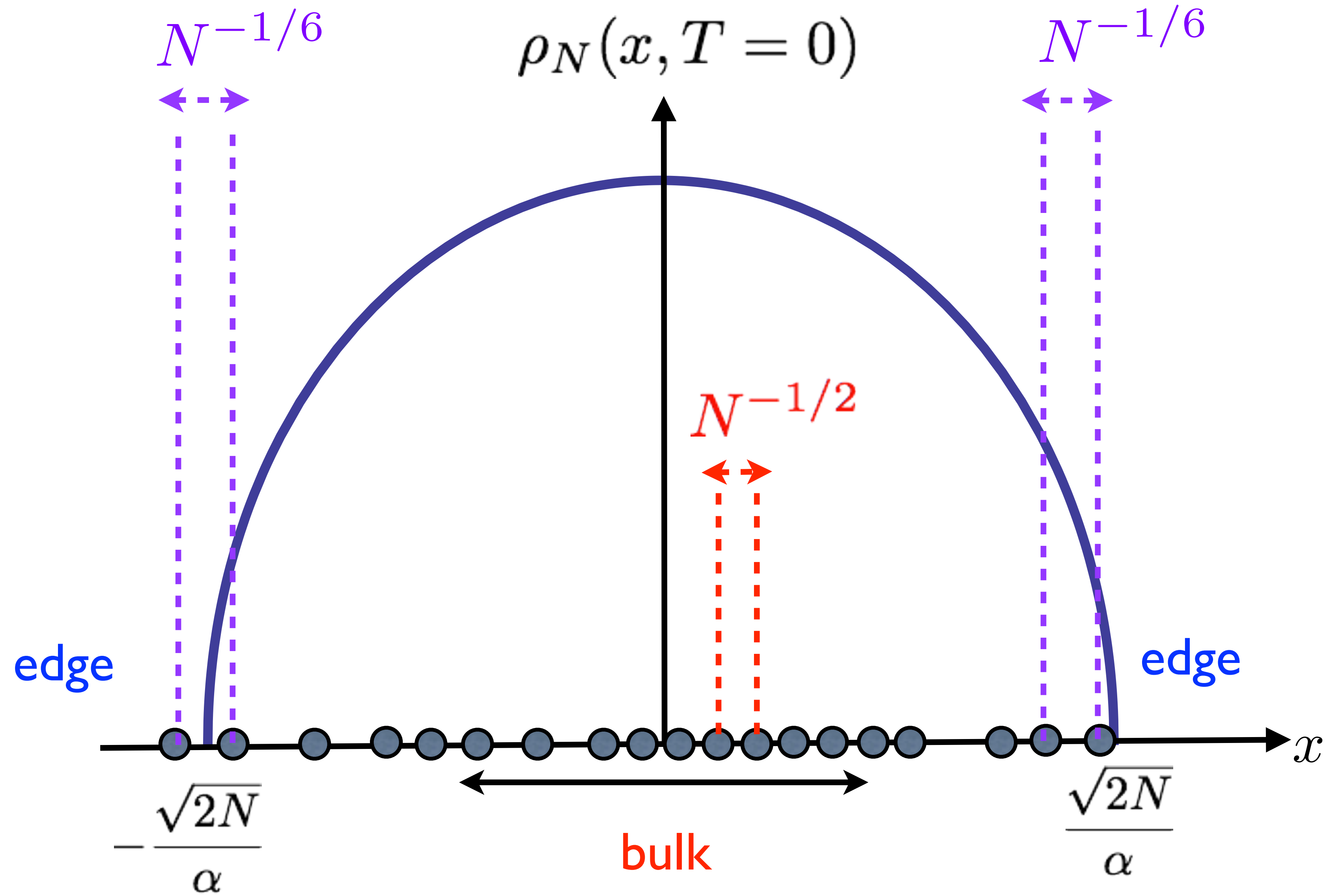


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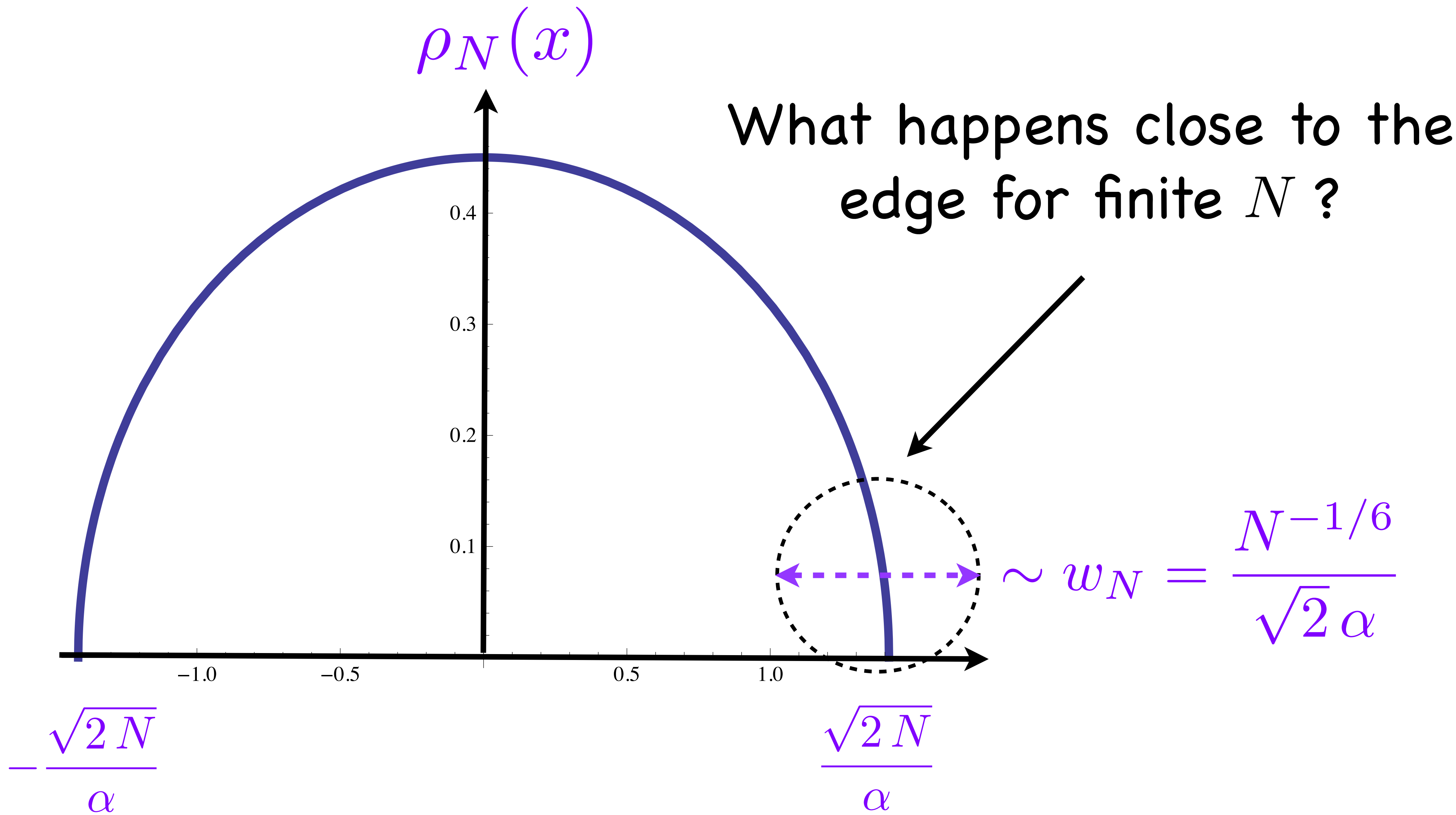
- edge interparticle distance: $\int_{r_{\rm edge}-\ell_{\rm edge}}^{r_{\rm edge}} \rho_N(x, T=0)\, dx \approx 1/N \Longrightarrow \ell_{\rm edge} \sim \frac{1}{\alpha} N^{-1/6}$

# Edge density for finite N at T=0



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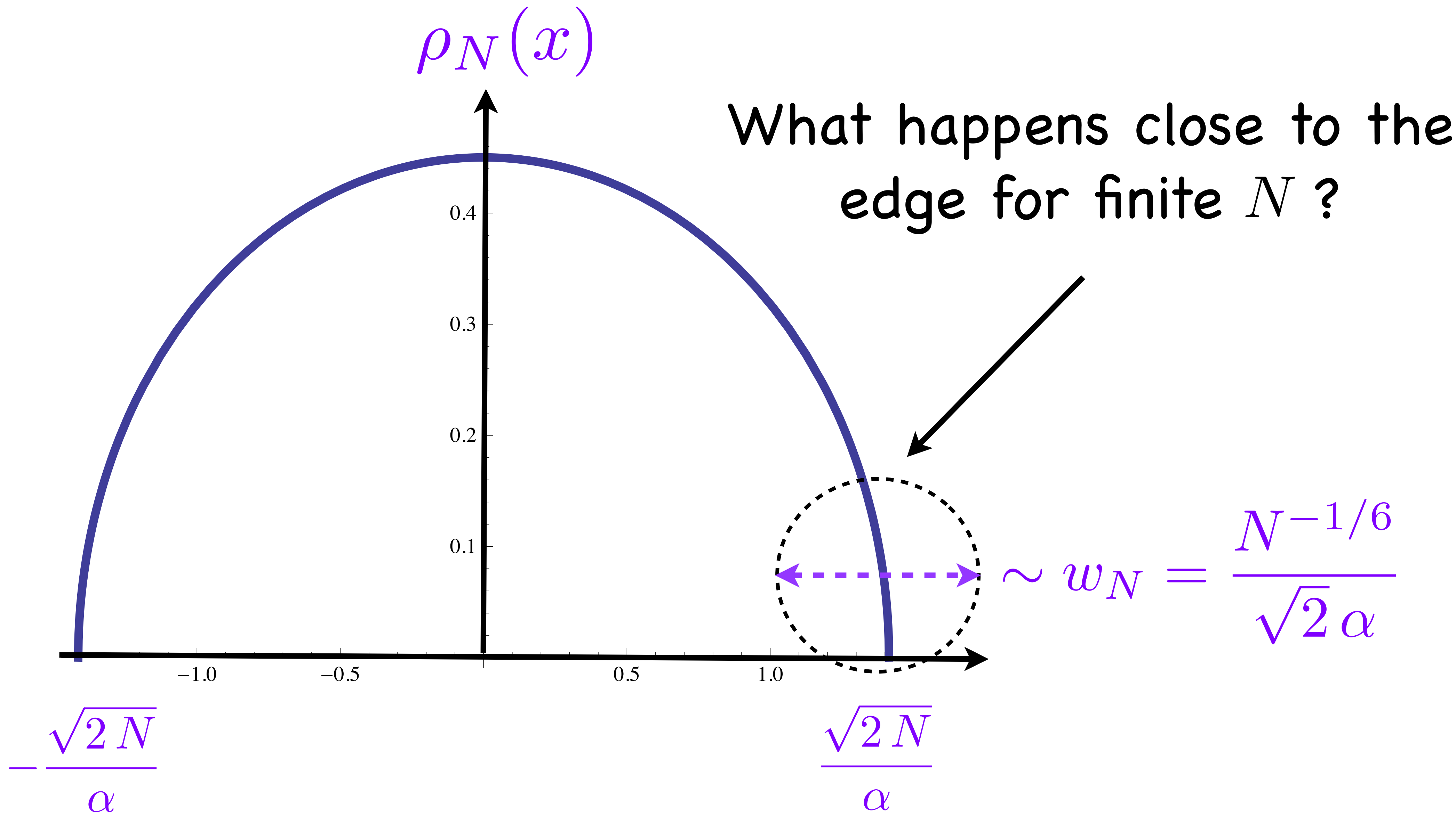


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# Random matrix theory ``comes to the rescue''

- Edge density of free fermions

Bowick, Brézin '91/Forrester '93

$$\rho_N(x) \approx \frac{1}{N w_N} F_1\left(\frac{x - \sqrt{2N}/\alpha}{w_N}\right)$$

with $w_N = \dfrac{N^{-1/6}}{\sqrt{2}\,\alpha}$ and $F_1(z) = [Ai'(z)]^2 - z[Ai(z)]^2$

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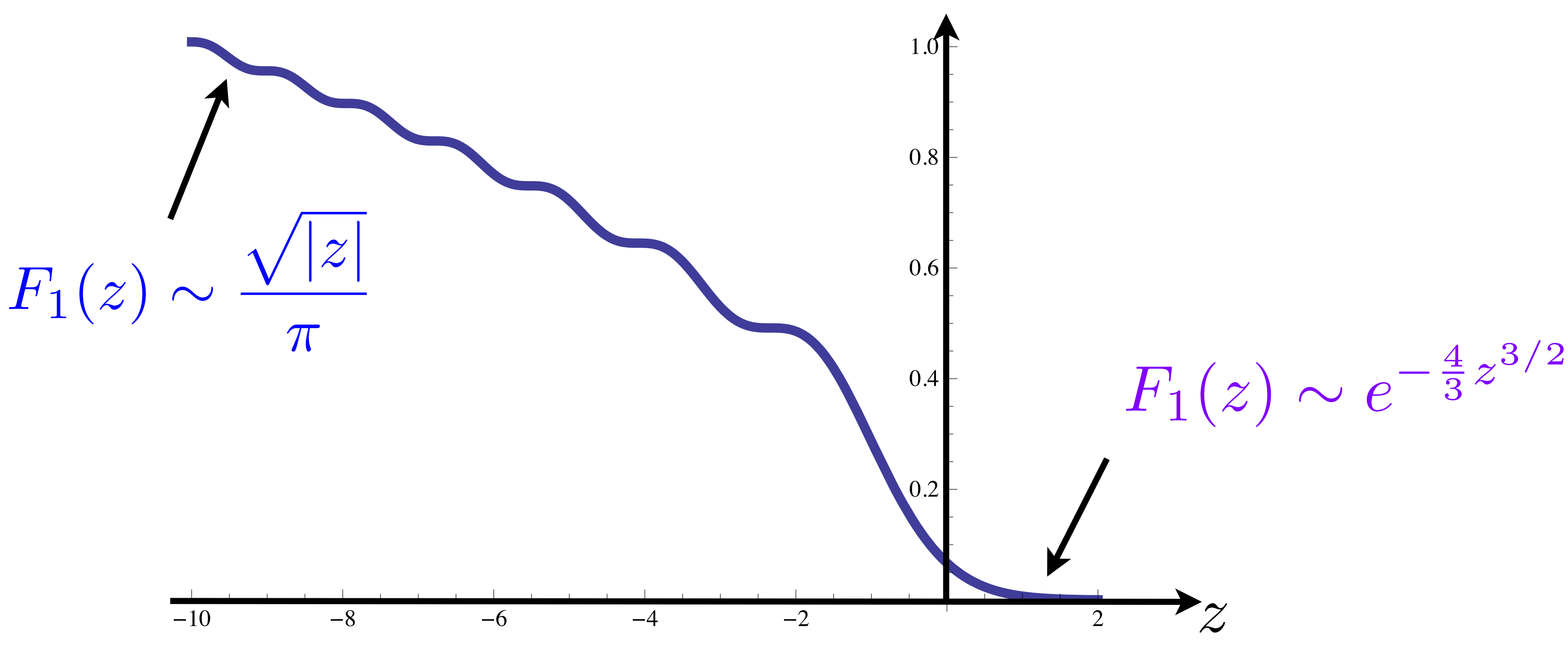

# Fermions in a 1d harmonic trap at T=0: kernel

- Higher order correlations

e.g., 2-point correlation function: $R_2(y,z) = \sum_{i \neq j} \langle \delta(y - x_i) \delta(z - x_j) \rangle$

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- Analogue of Wick 's theorem: $K_N(x,y) = \langle \Phi_{\rm gs}|\Psi^\dagger(x)\Psi(y)|\Phi_{\rm gs}\rangle$

# Limiting form of the kernel for trapped fermions at T=0

- Bulk limit: when $x$ & $y$ are far from the edge and

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- Edge scaling limit: for $x$ & $y$ close to the edge $r_{\rm edge} = \sqrt{2N}/\alpha$

$$K_N(x,y) \approx \frac{1}{w_N}\mathcal{K}_{\rm edge}\left(\frac{x-r_{\rm edge}}{w_N}, \frac{y-r_{\rm edge}}{w_N}\right) \; , \; w_N = \frac{N^{-1/6}}{\sqrt{2}\alpha}$$

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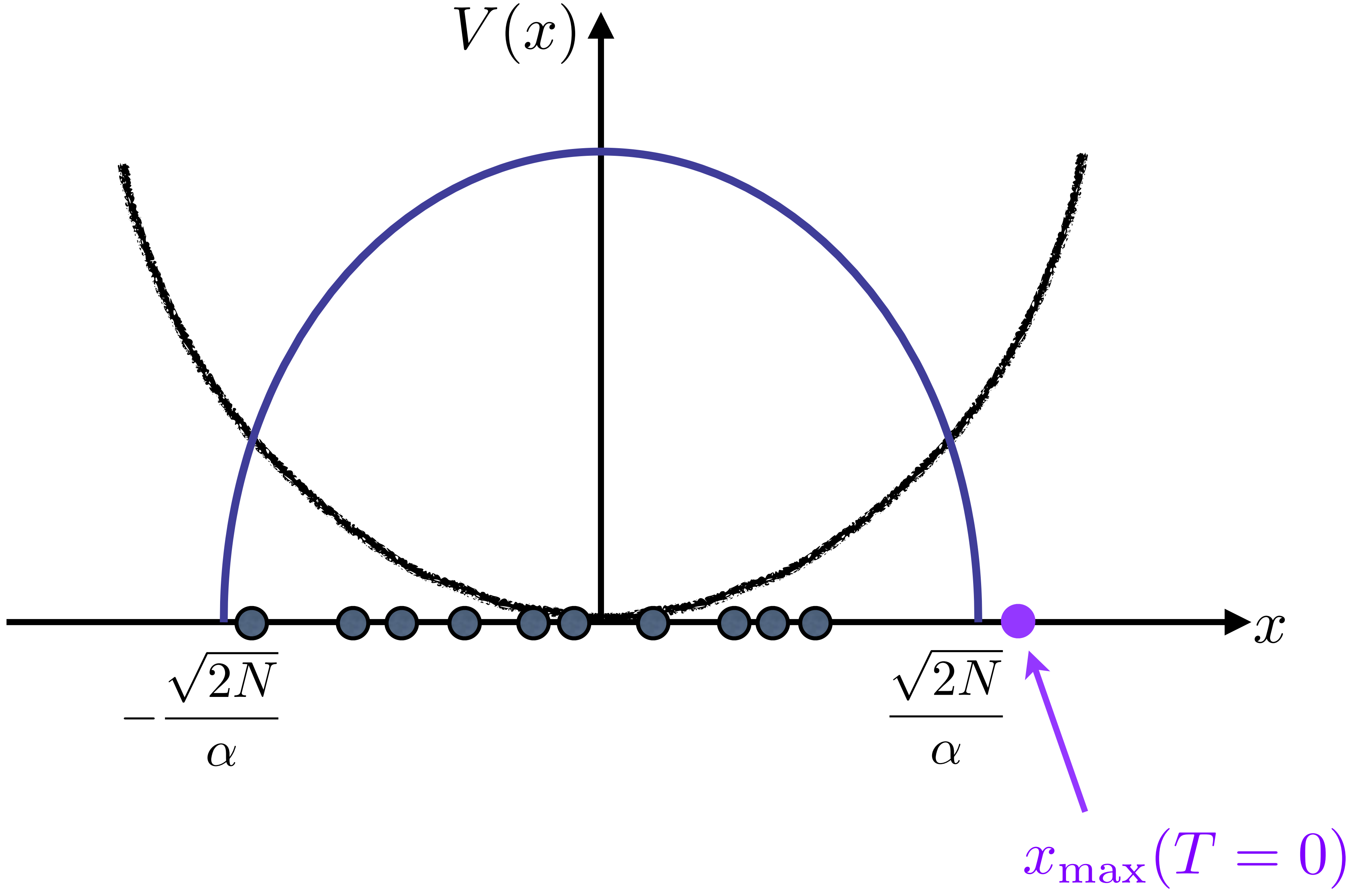
Airy-kernel

# Position of the rightmost fermion at T=0



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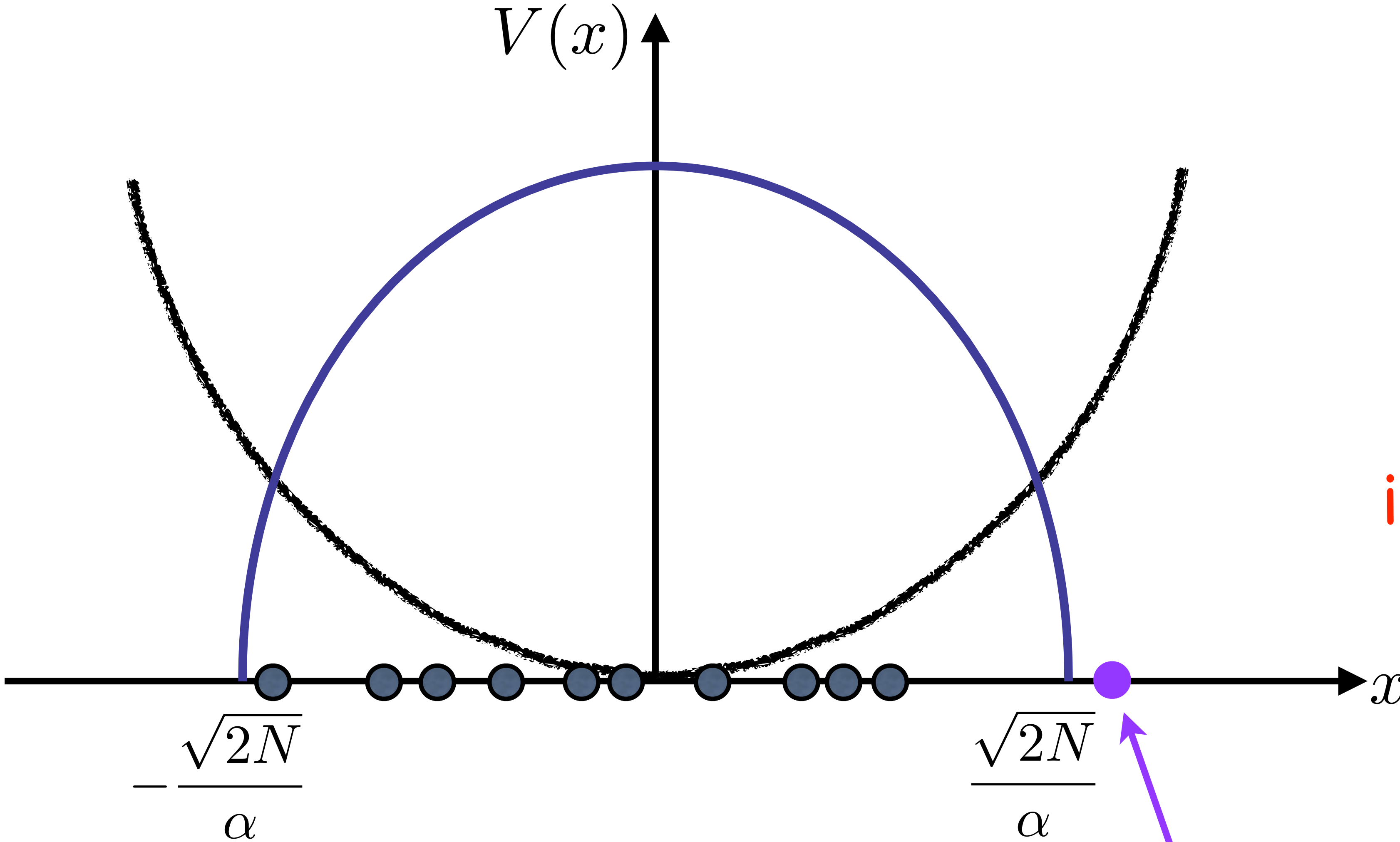


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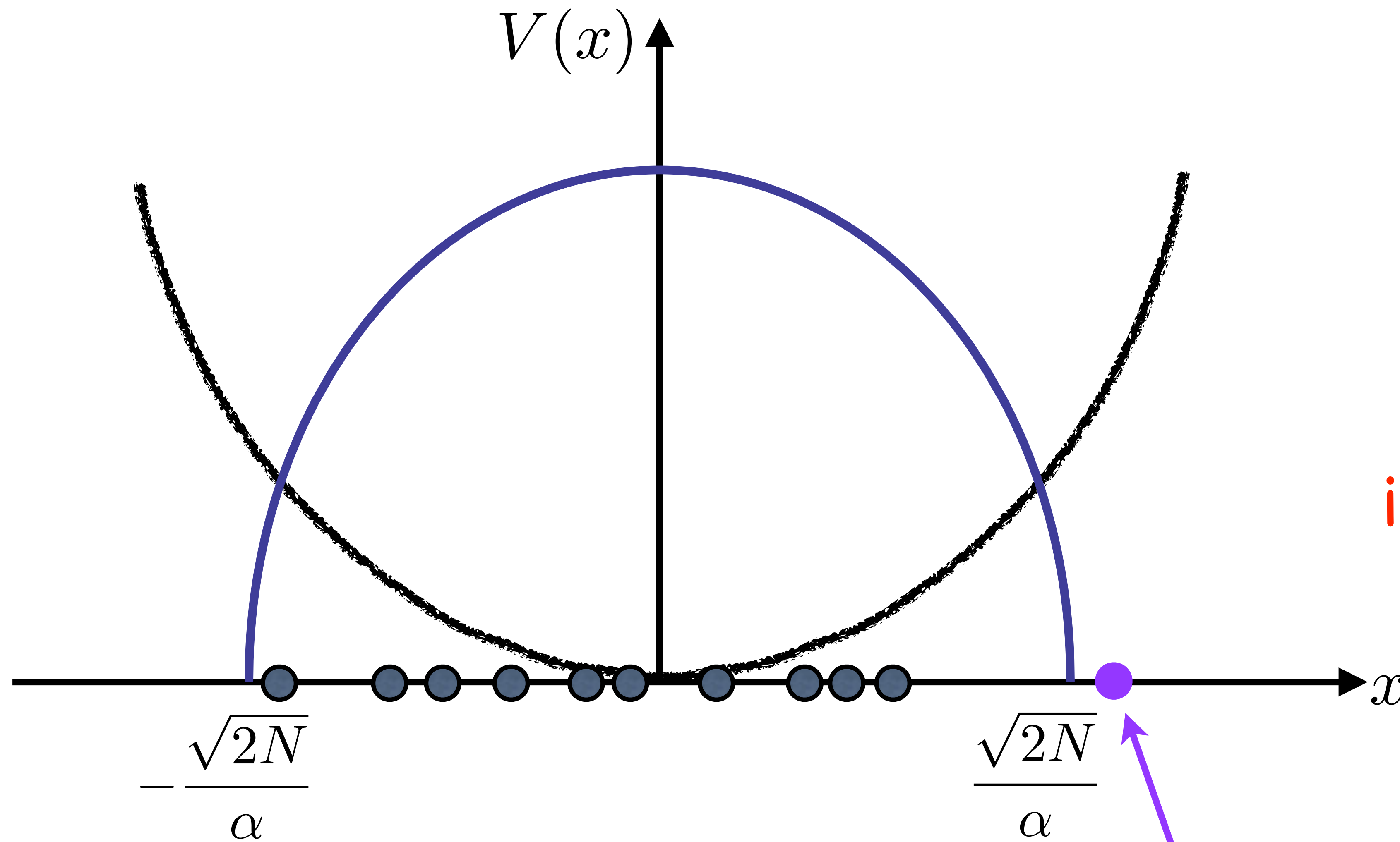


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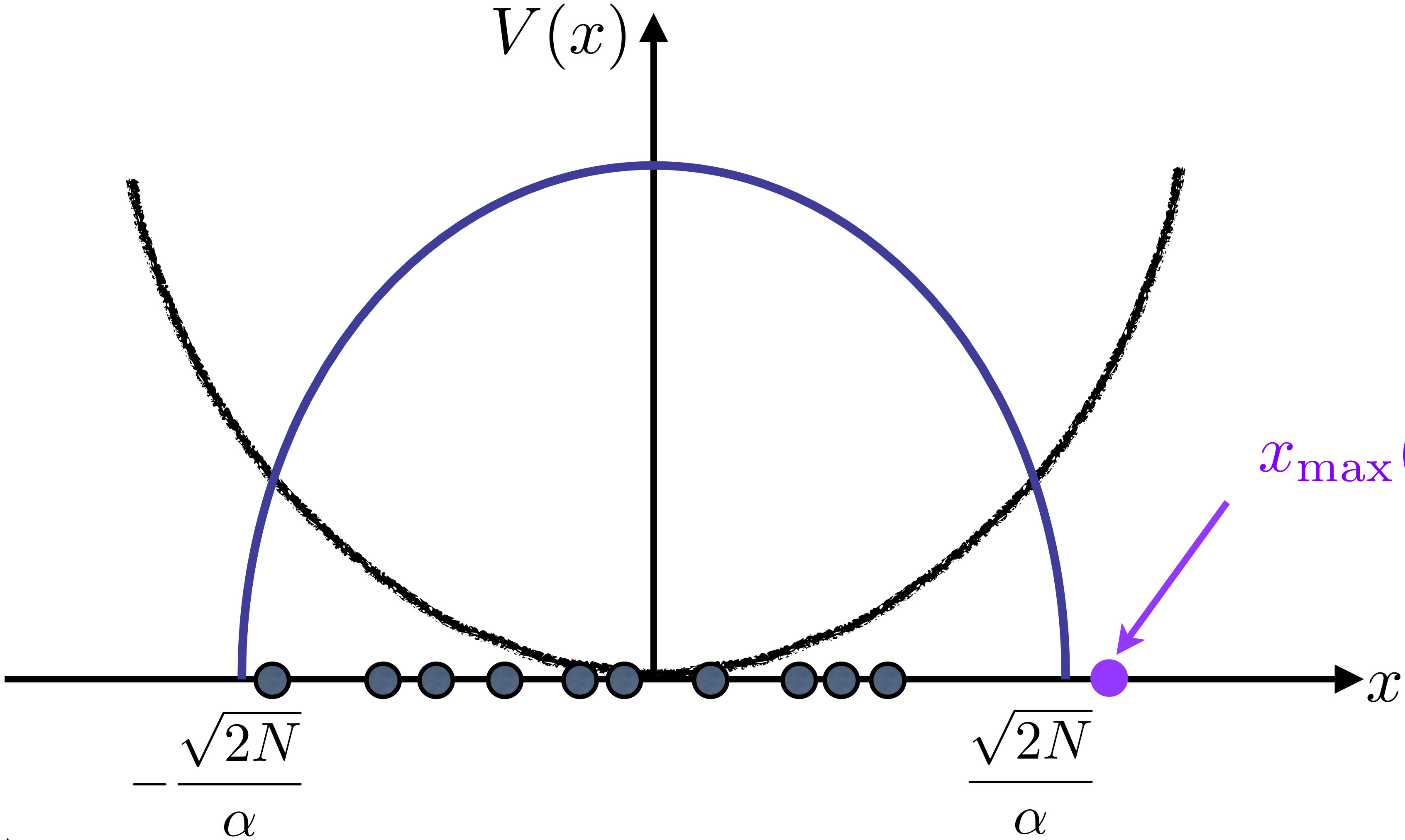


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⇒ fluctuations of $x_{\max}(T=0)$ are governed by the Tracy-Widom distribution for GUE

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Fredholm determinant

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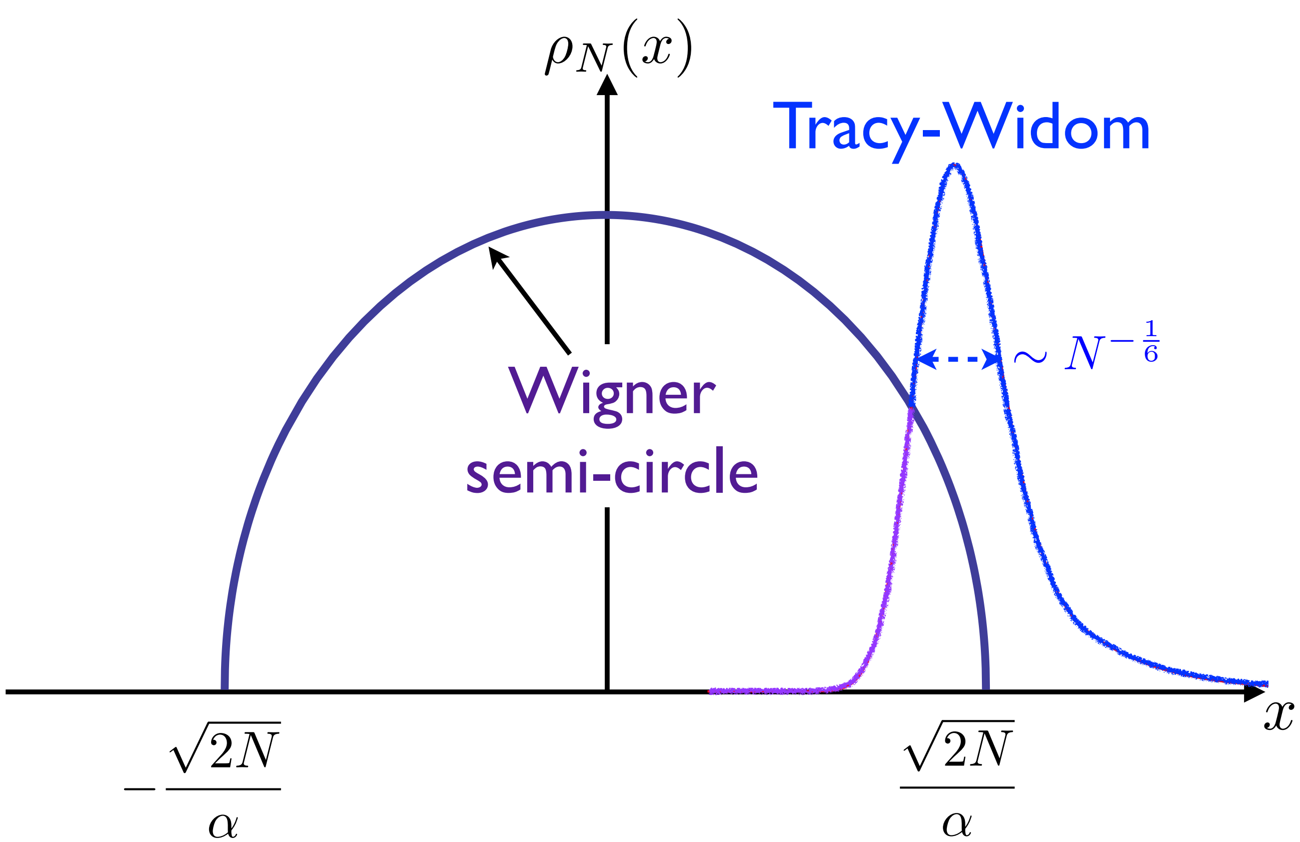
# Largest (top) eigenvalue of random matrices

- $J_{ij}$ : complex Hermitian $N \times N$ Gaussian random matrix

- Recent excitements in statistical physics and mathematics on

$$\lambda_{\max} = \max_{1 \le i \le N} \lambda_i : \text{ largest eigenvalue of } J$$

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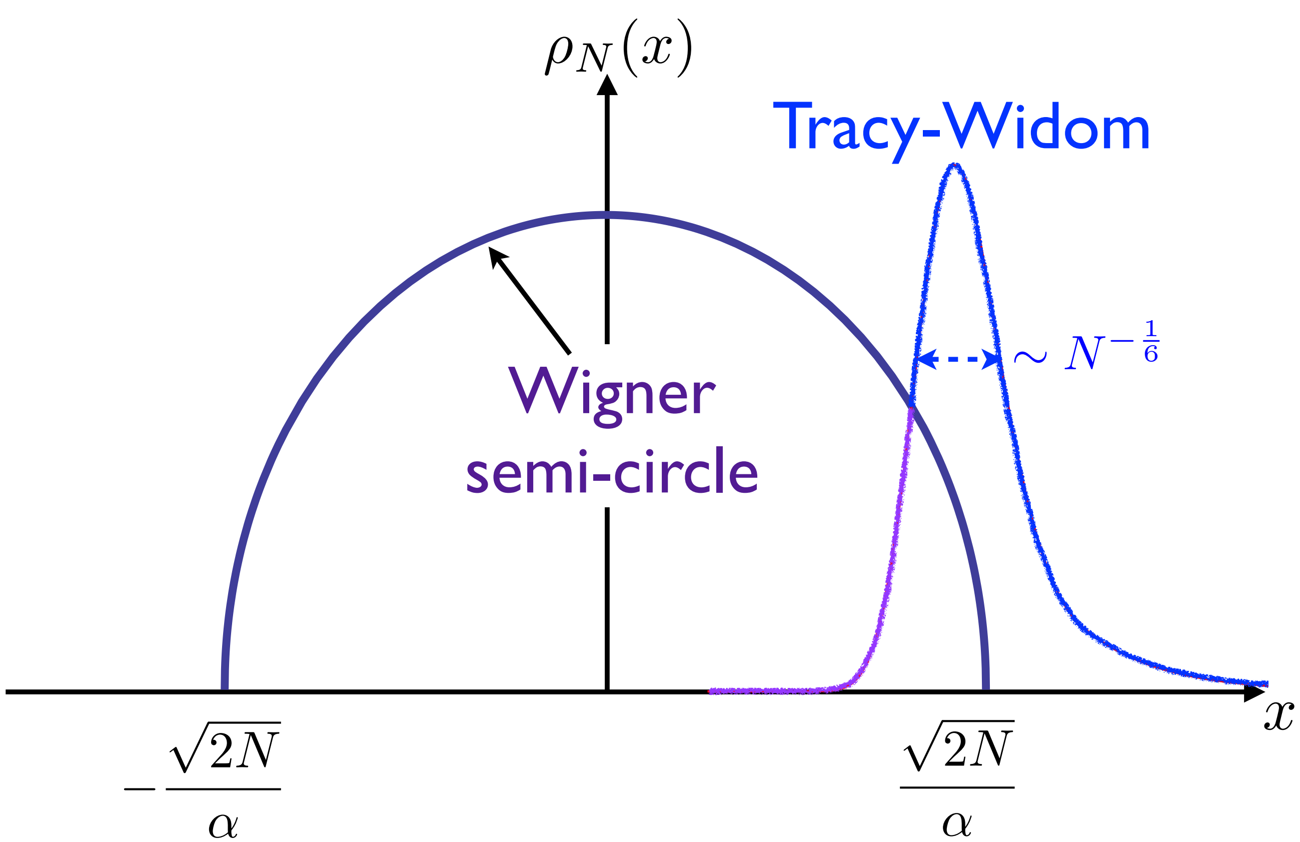


Typical fluctuations:

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KPZ equation, directed polymer, random permutation, sequence alignment,...

- Free fermions provide (one of) the simplest physical systems where the Tracy-Widom distribution can be observed

What happens at finite temperature

$T > 0$ ?

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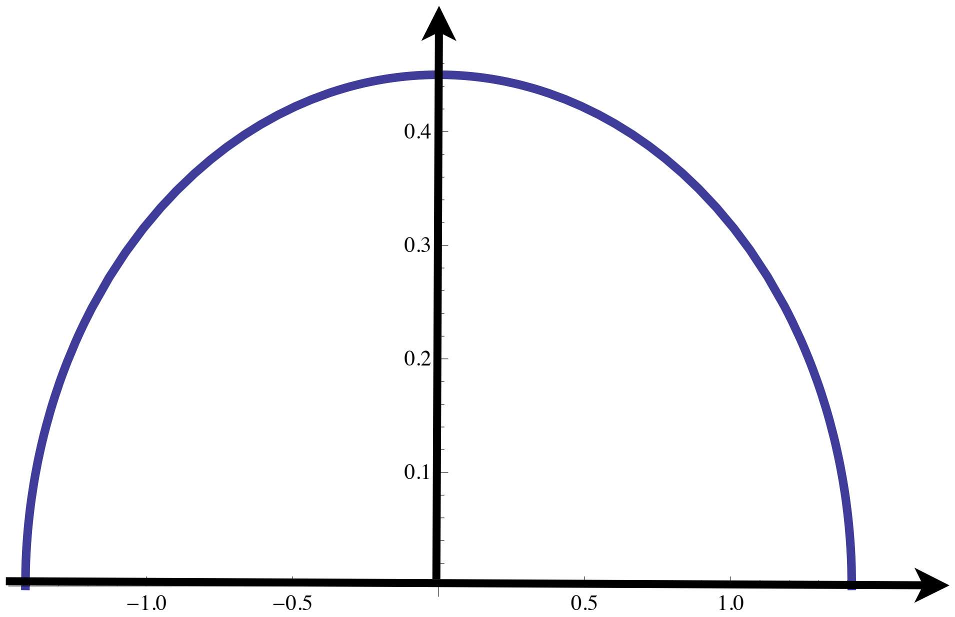


$$\rho_N(x, T\to 0) \approx \frac{\alpha}{\sqrt{N}} f_W\left(\frac{\alpha\, x}{\sqrt{N}}\right) \;,$$

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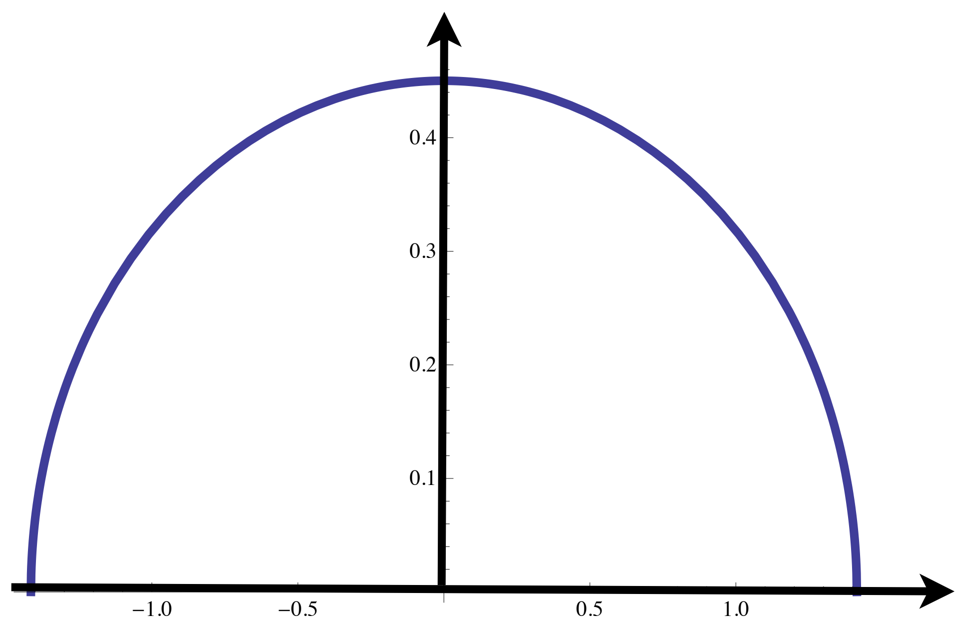


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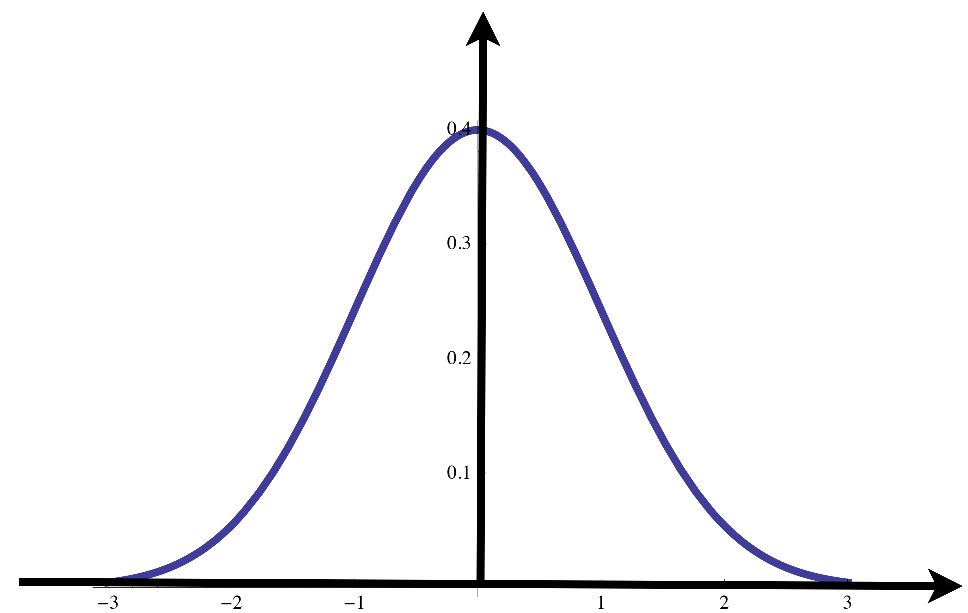


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Interpolation at finite T ?

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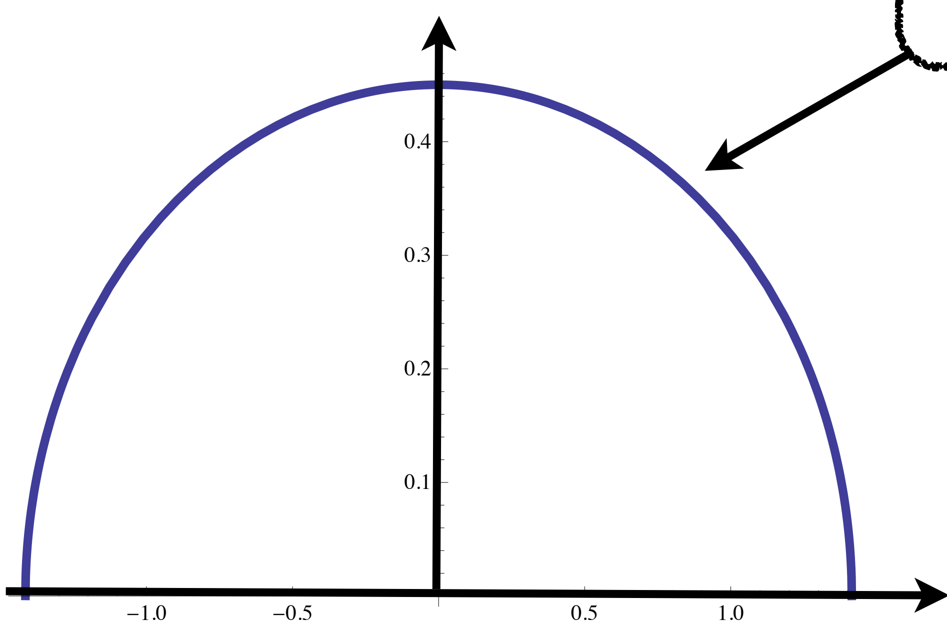


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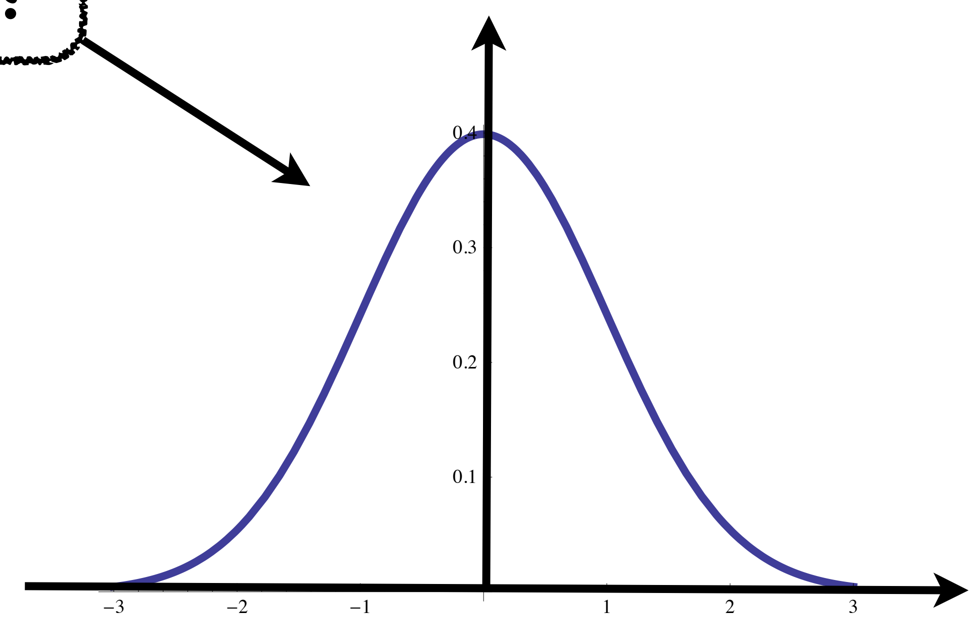


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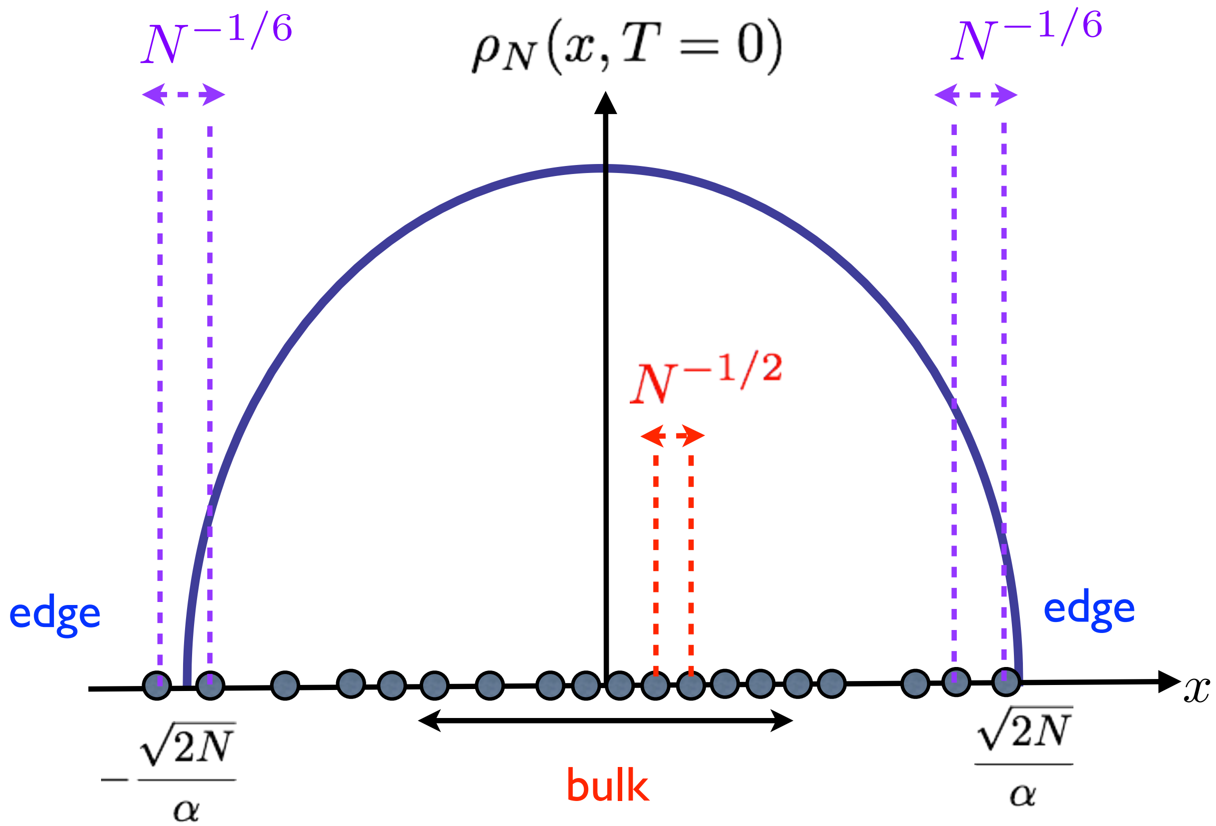


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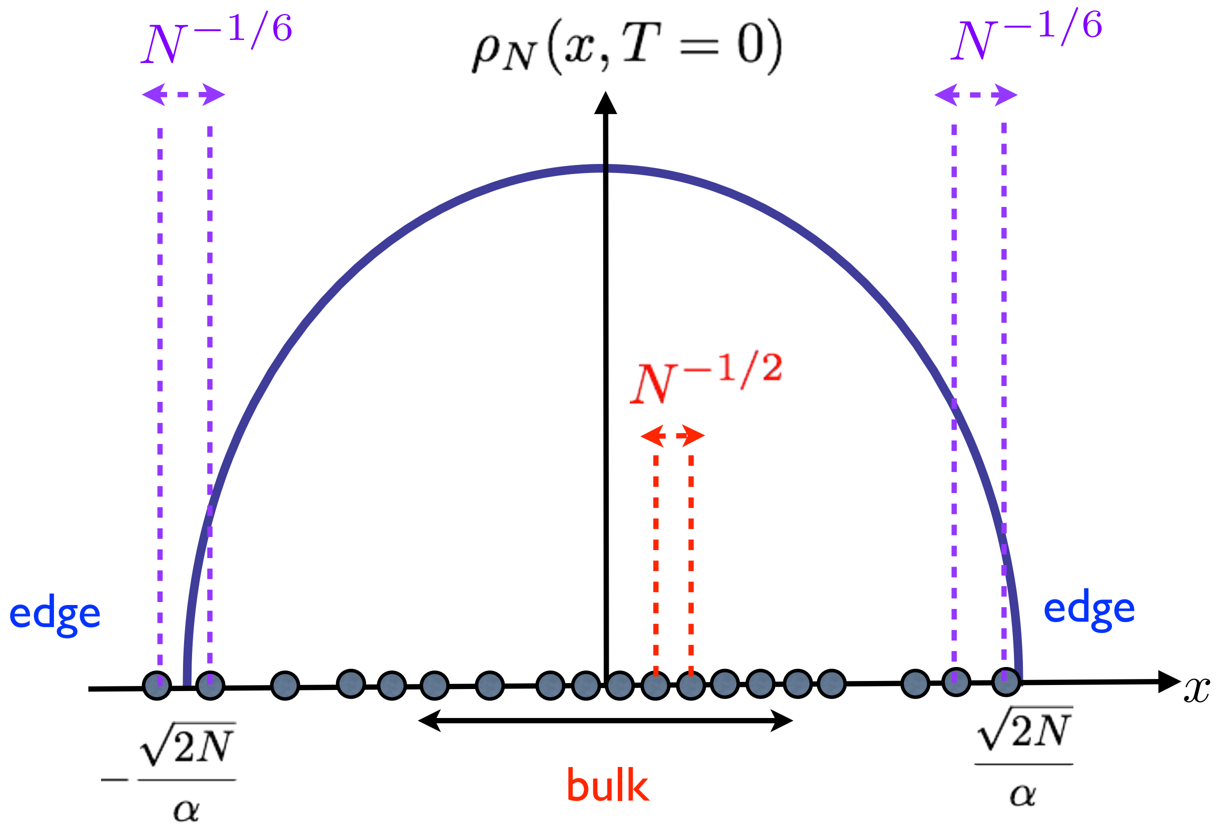


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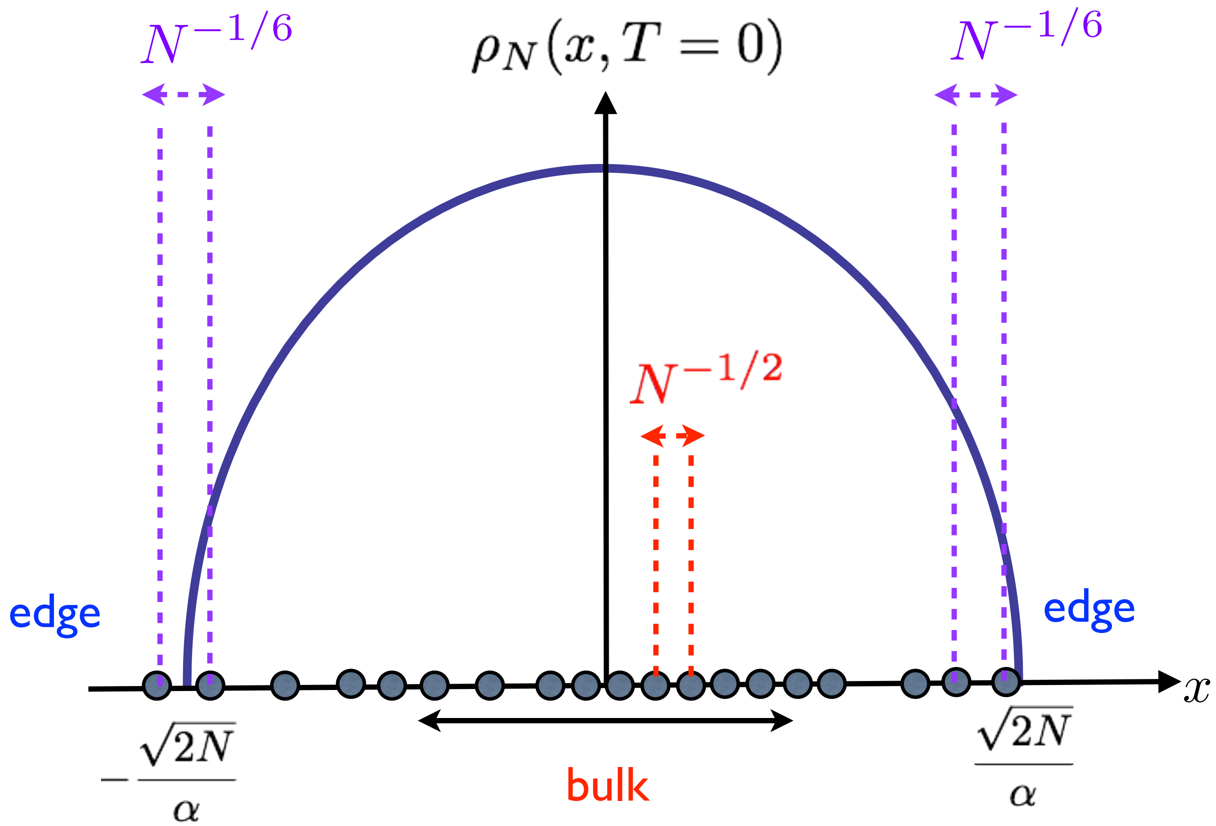


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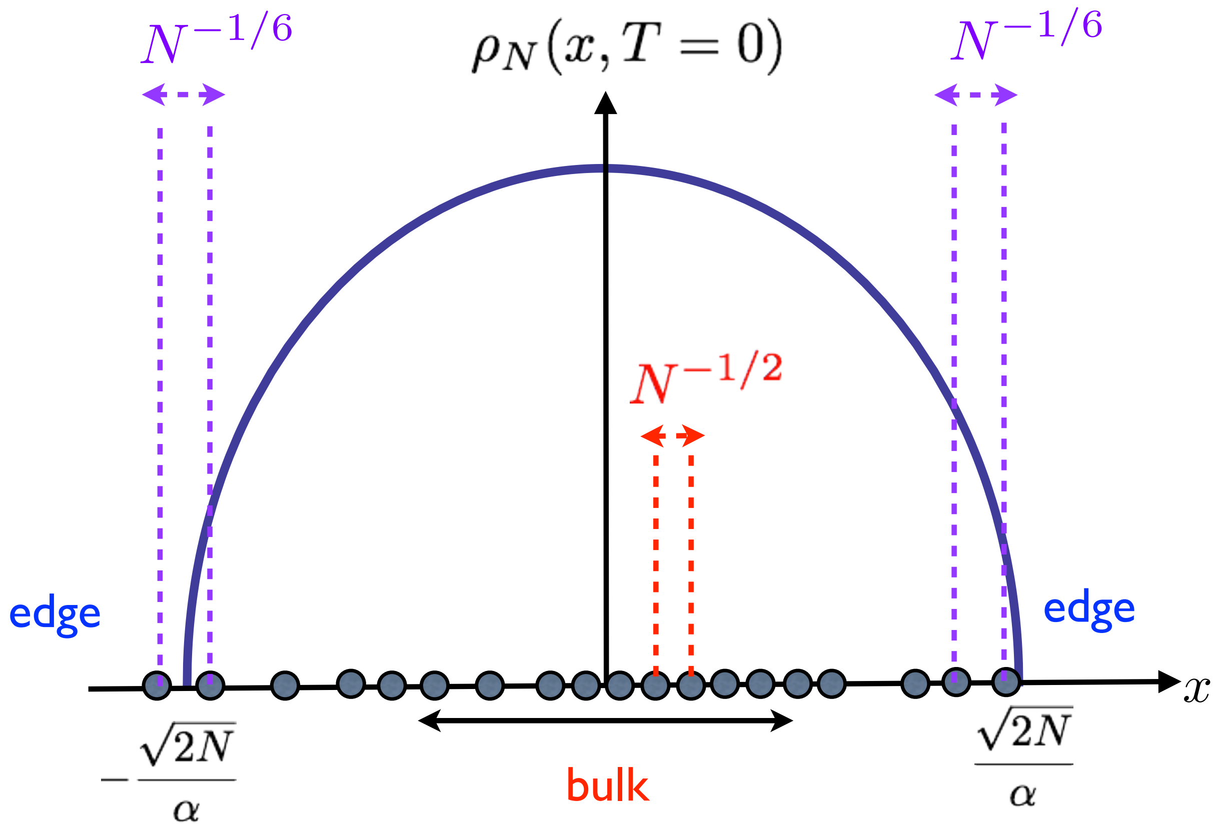


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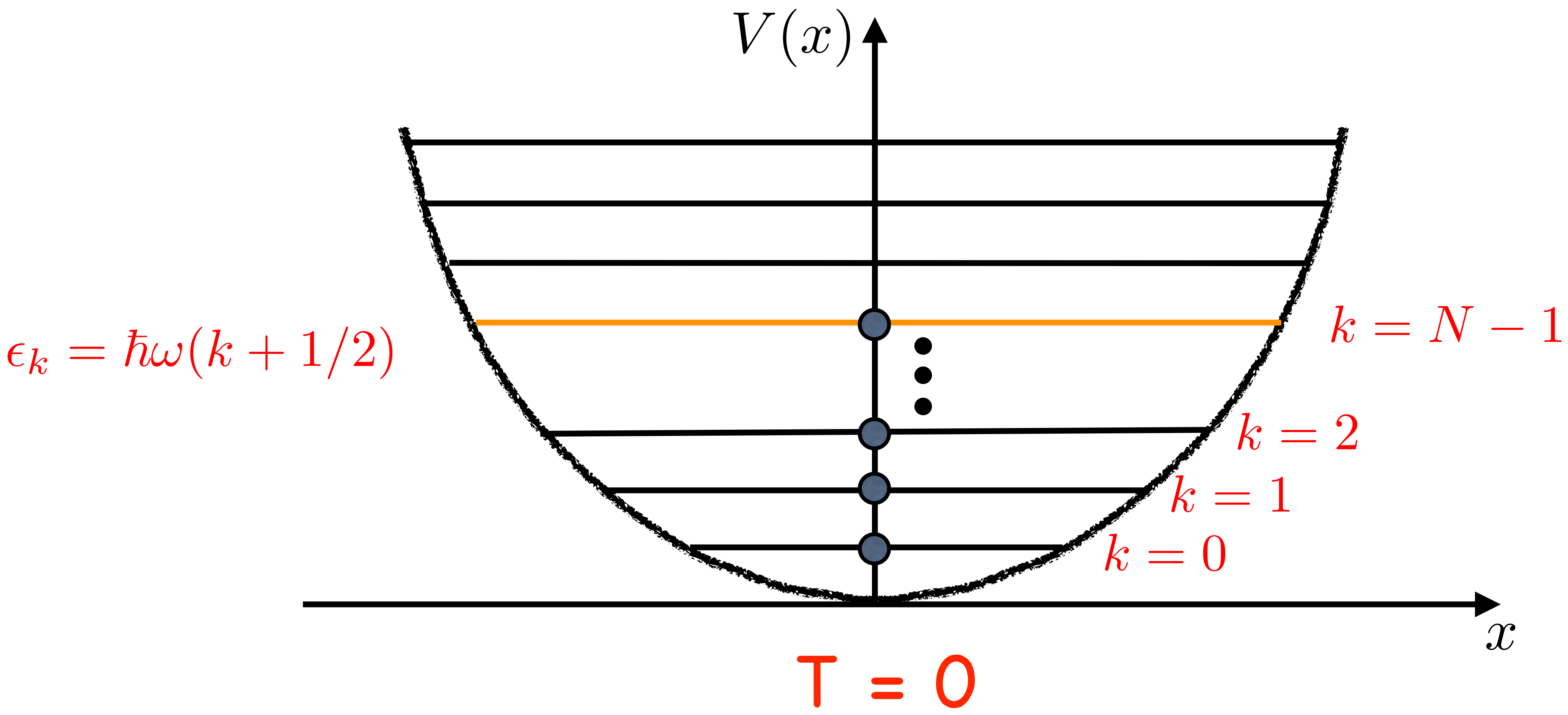

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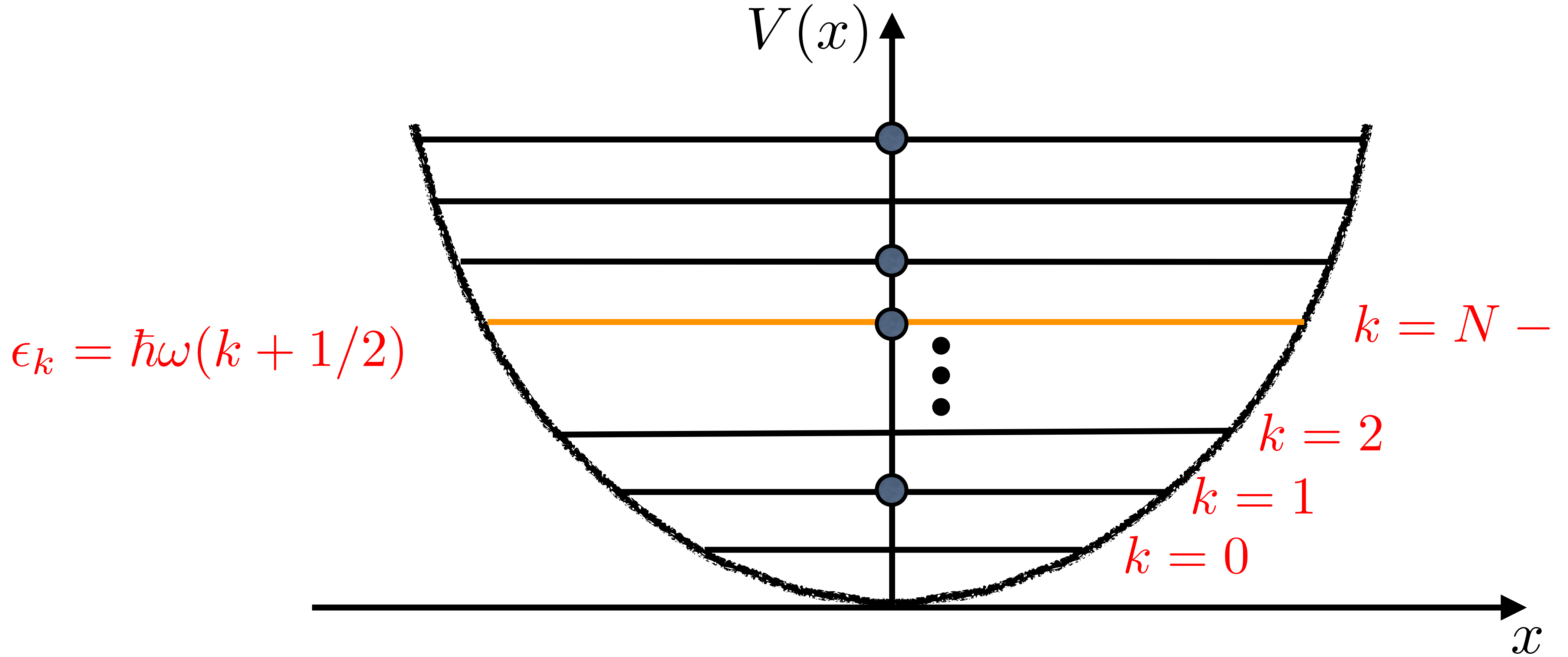


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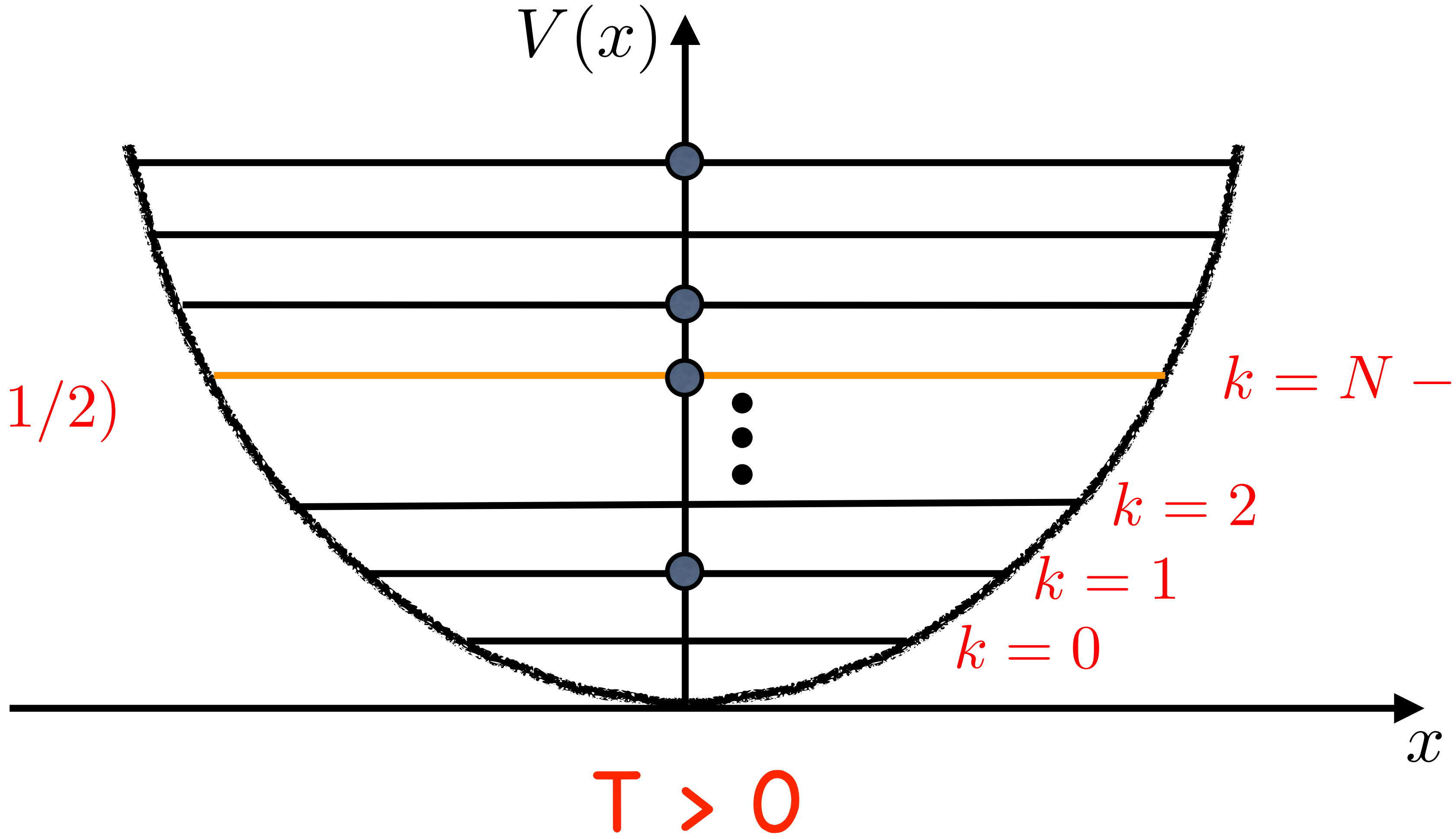


- Probability density function (PDF) of the positions $x_i's$

$$P_{\rm joint}(x_1, \cdots x_N) = \frac{1}{N! Z_N(\beta)} \sum_{k_1 < \cdots < k_N} \left[ \det_{1 \leq i,j \leq N} (\varphi_{k_i}(x_j)) \right]^2 e^{-\beta(\epsilon_{k_1} + \cdots + \epsilon_{k_N})}$$

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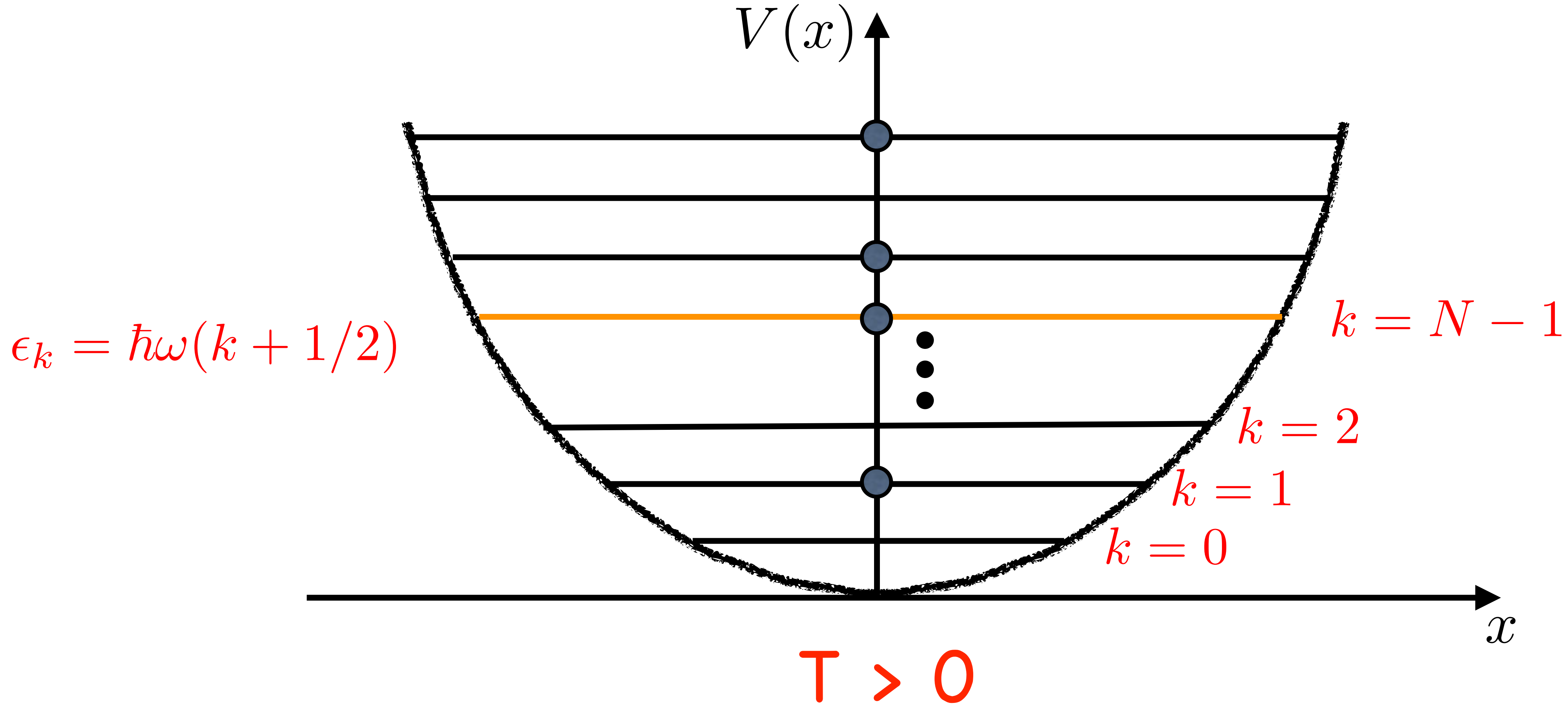


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# Time-periodic OU process

- Ornstein-Uhlenbeck (OU) process starting at $X_0 = x_0$

$$dX_\tau = -\mu_0 \, X_\tau \, d\tau + dB_\tau$$

$$\mathbb{P}(X_\tau \in dx | X_{\tau_0} = x_0) = P_{\mathrm{OU}}(x, \tau | x_0, \tau_0) \, dx$$

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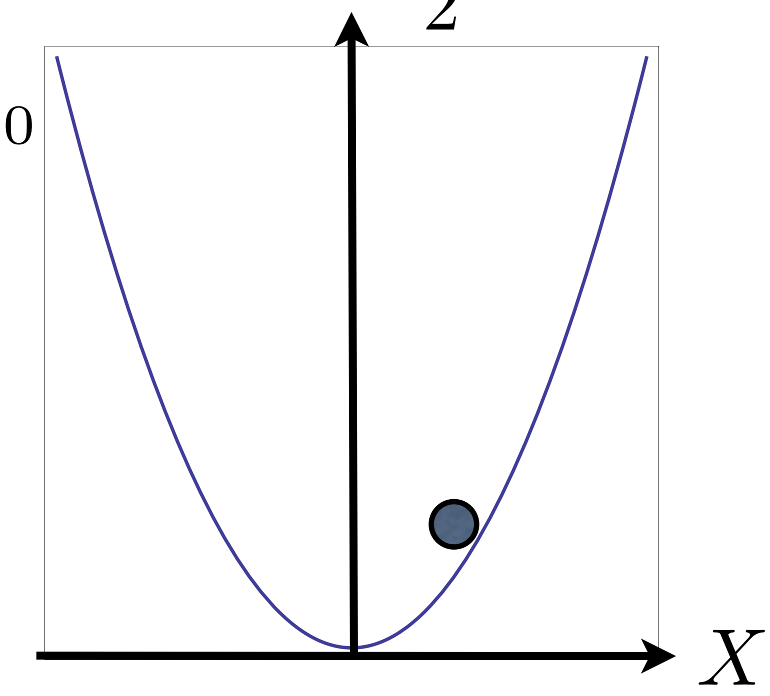

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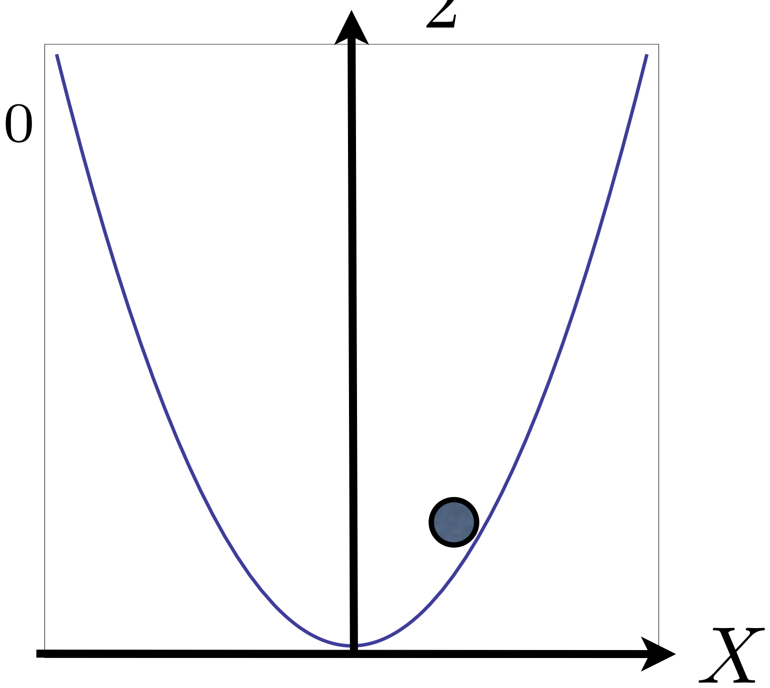

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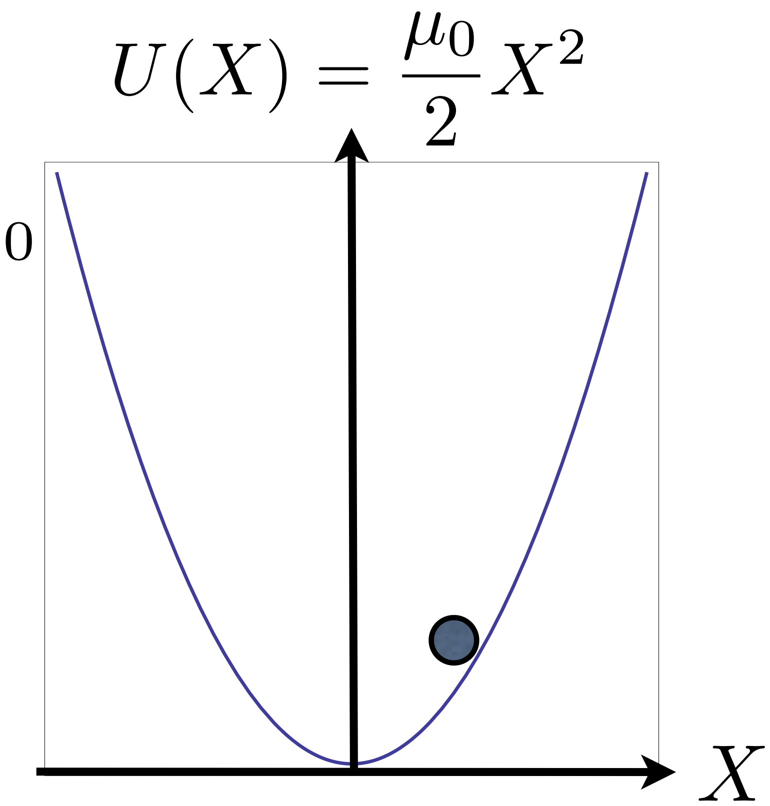


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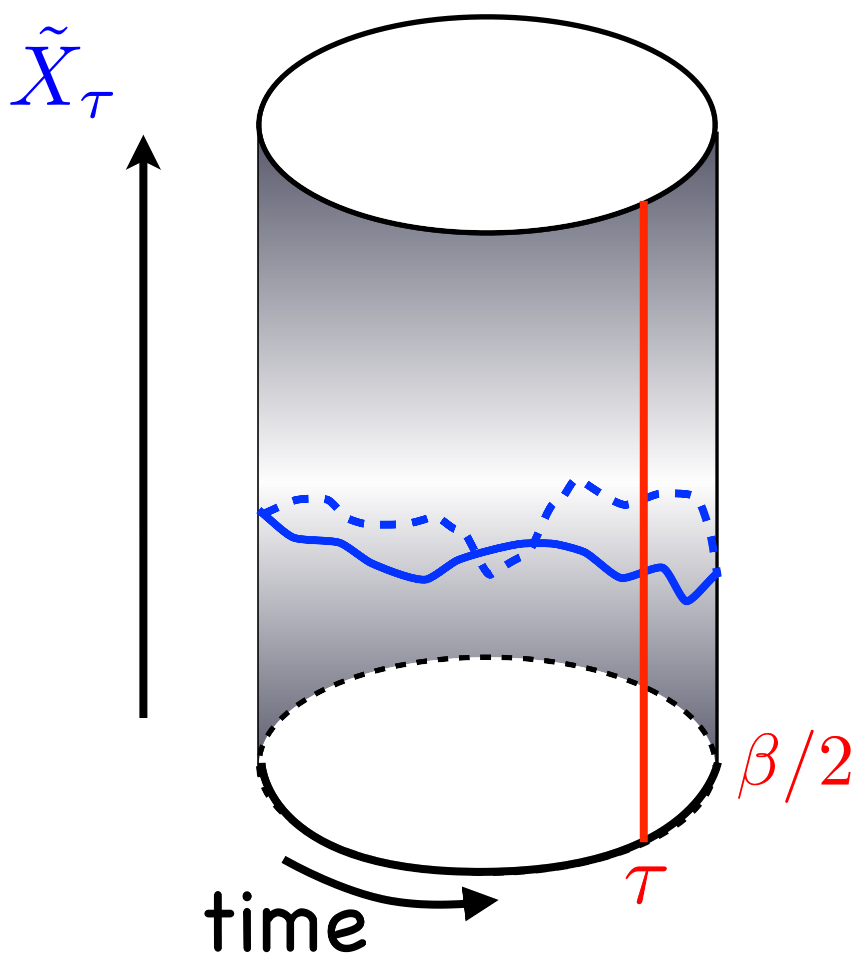

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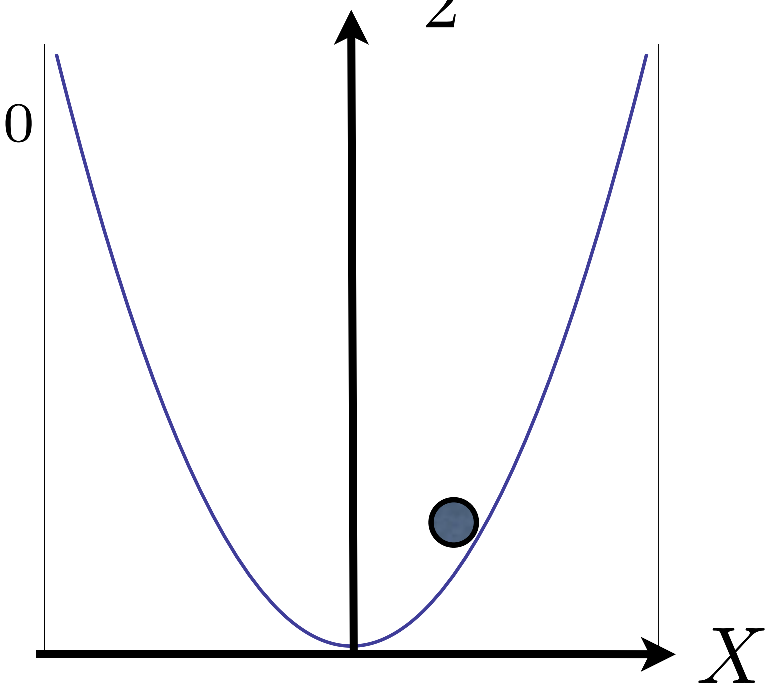


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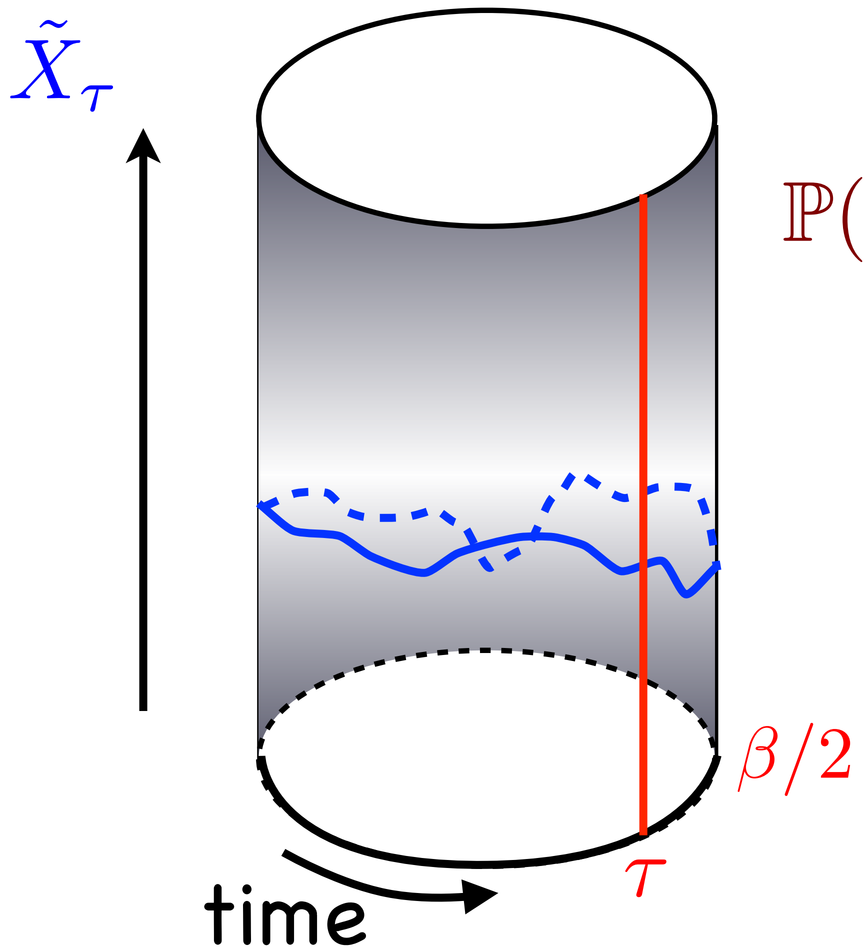


$$\mathbb{P}(\tilde{X}_\tau \in dx) = P_\beta(x) dx \ , \ P_\beta(x) = \frac{1}{Z_1} P_{\text{OU}}(x, \beta | x, 0)$$

$$P_\beta(x) = \sqrt{\frac{\mu_0}{\pi} \tanh\left(\frac{\beta \mu_0}{2}\right)} \, e^{-\mu_0 \tanh\left(\frac{\beta\mu_0}{2}\right) x^2}$$

independently of $\tau$

# Single quantum particle at finite temperature

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- PDF of the position of the particle at finite temperature $T = 1/\beta$

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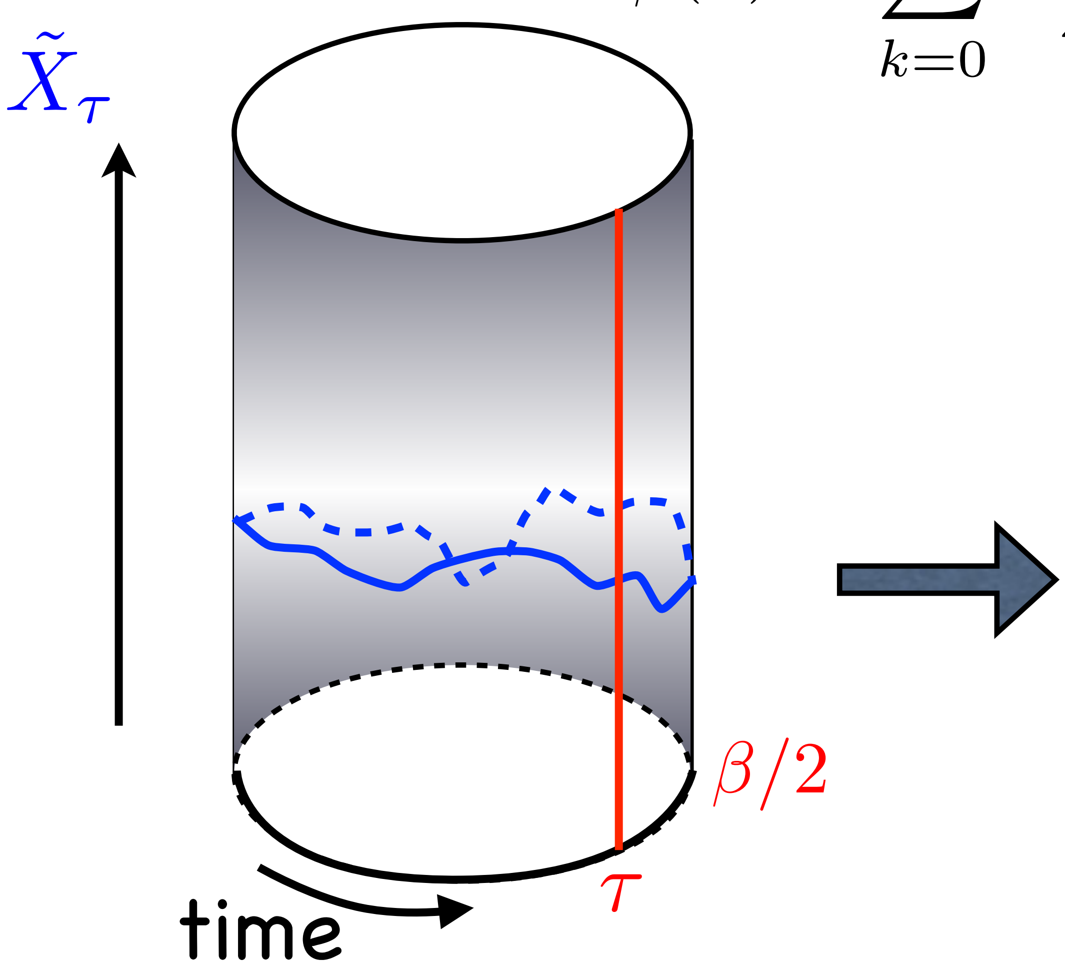


time periodic OU process

# N fermions at finite temperature

- PDF of the positions of the particle at finite temperature $T = 1/\beta$

$$P_\beta(x_1, \cdots, x_N) = \frac{1}{Z_N(\beta)} \sum_E |\psi_E(x_1, \cdots, x_N)|^2 e^{-\beta E}$$

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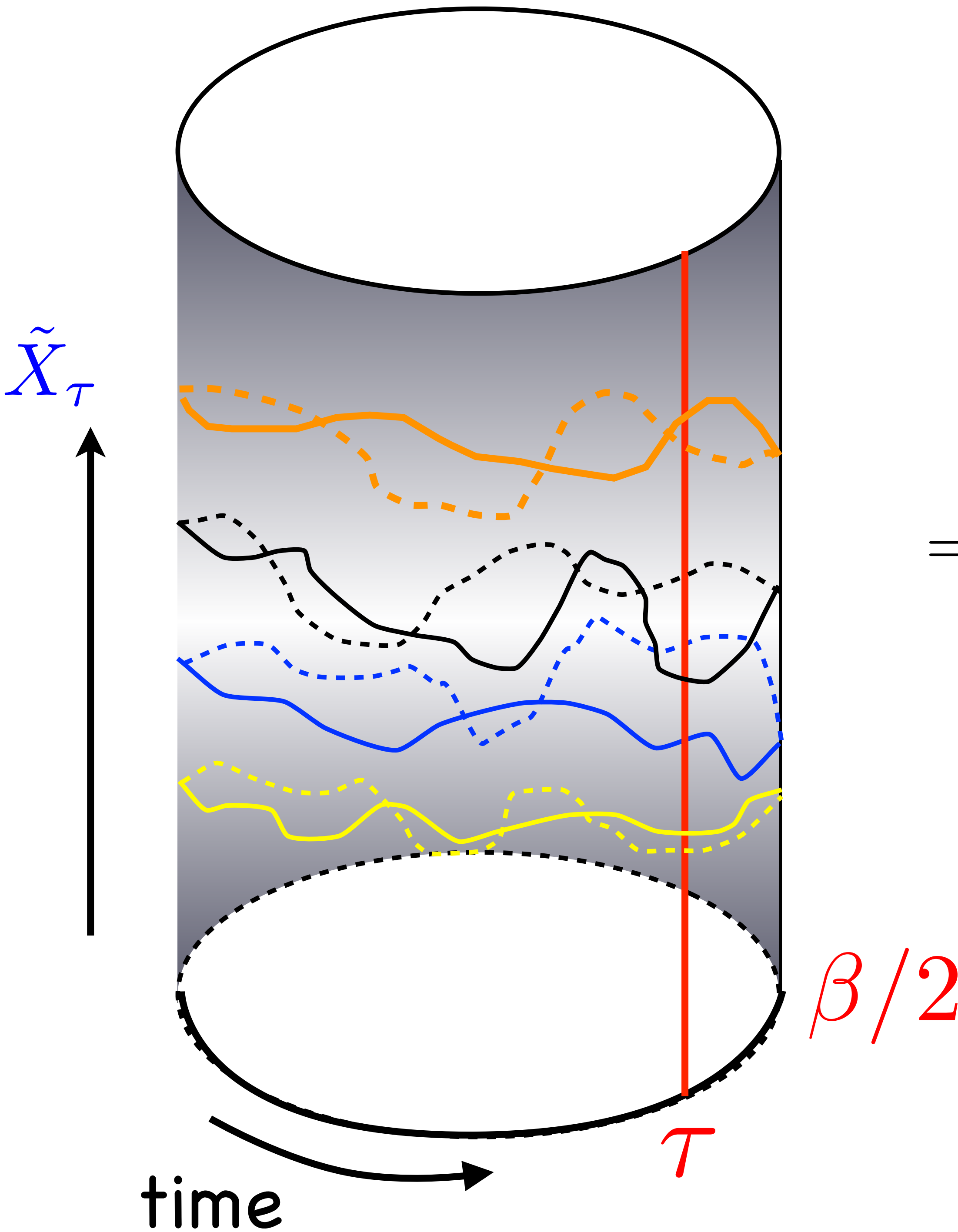


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non-intersecting time periodic OU process

Le Doussal, Majumdar, G. S., Ann. Phys. '17

# Correlation kernel for $N$ free fermions at T > 0

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number of particles $N$ is fixed

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$n$-point correlation function $R_n(x_1, \cdots, x_n) \approx \det_{1 \leq i,j \leq n} K_\mu(x_i, x_j)$

$$K_\mu(x, x') = \sum_{k=0}^{\infty} \frac{\varphi_k(x)\varphi_k(x')}{e^{\beta(\epsilon_k - \mu)} + 1}$$

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see also Moshe, Neuberger, Shapiro '94/Johansson '07

# Average density of free fermions at T > 0

$$\rho_N(x,T) = \frac{1}{N}\sum_{i=1}^{N}\langle\delta(x-x_i)\rangle$$

- Two natural dimensionless variables

$$y = \frac{E_F}{T} = \frac{N\hbar\omega}{T} \qquad \text{and} \qquad z = x\sqrt{\frac{m\omega^2}{2T}}$$

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See also Local Density (or Thomas-Fermi) Approx. in the literature on fermions

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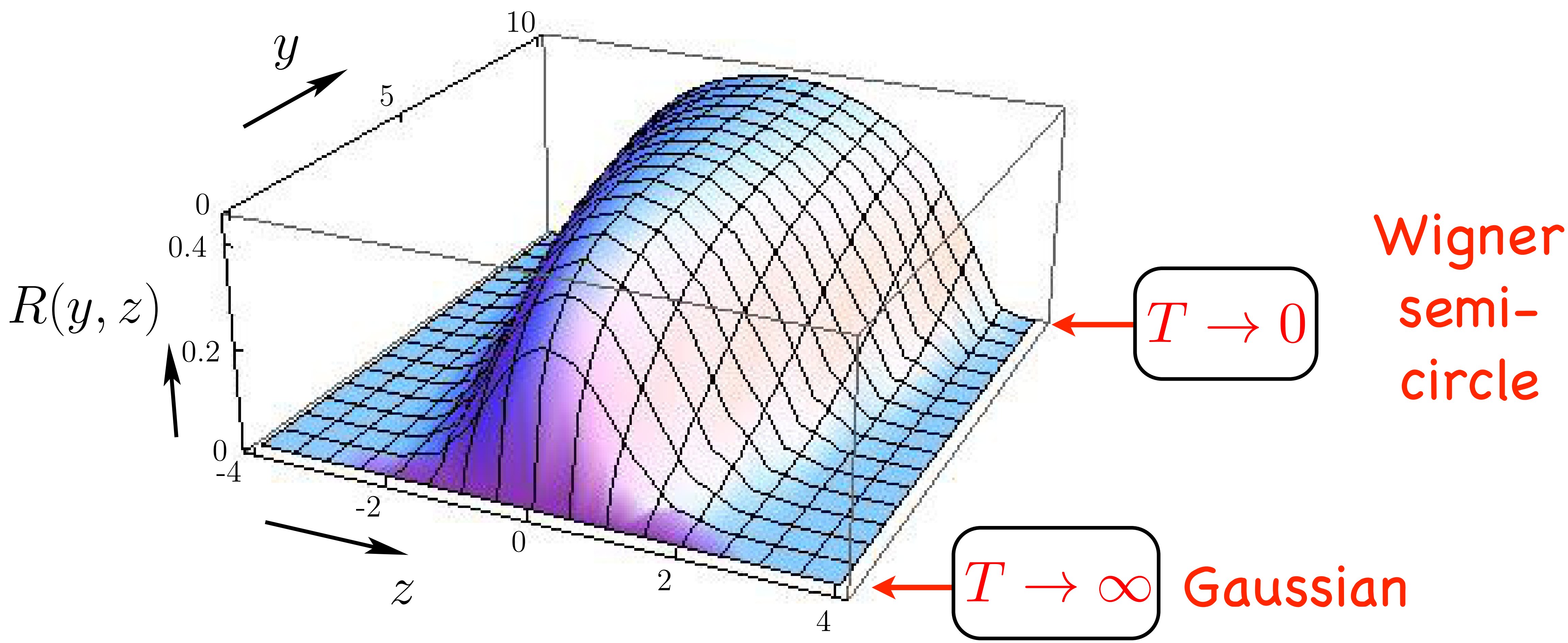

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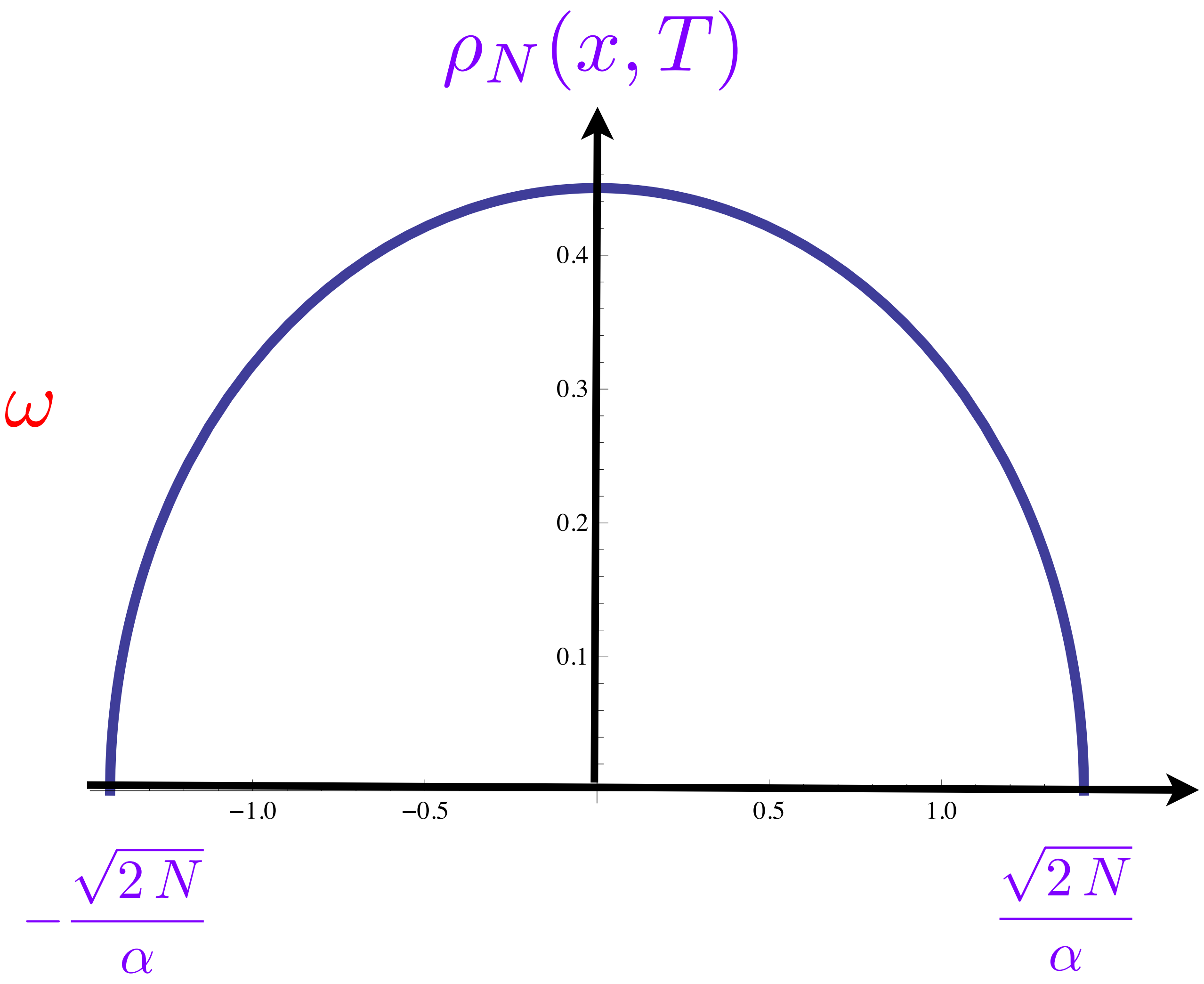

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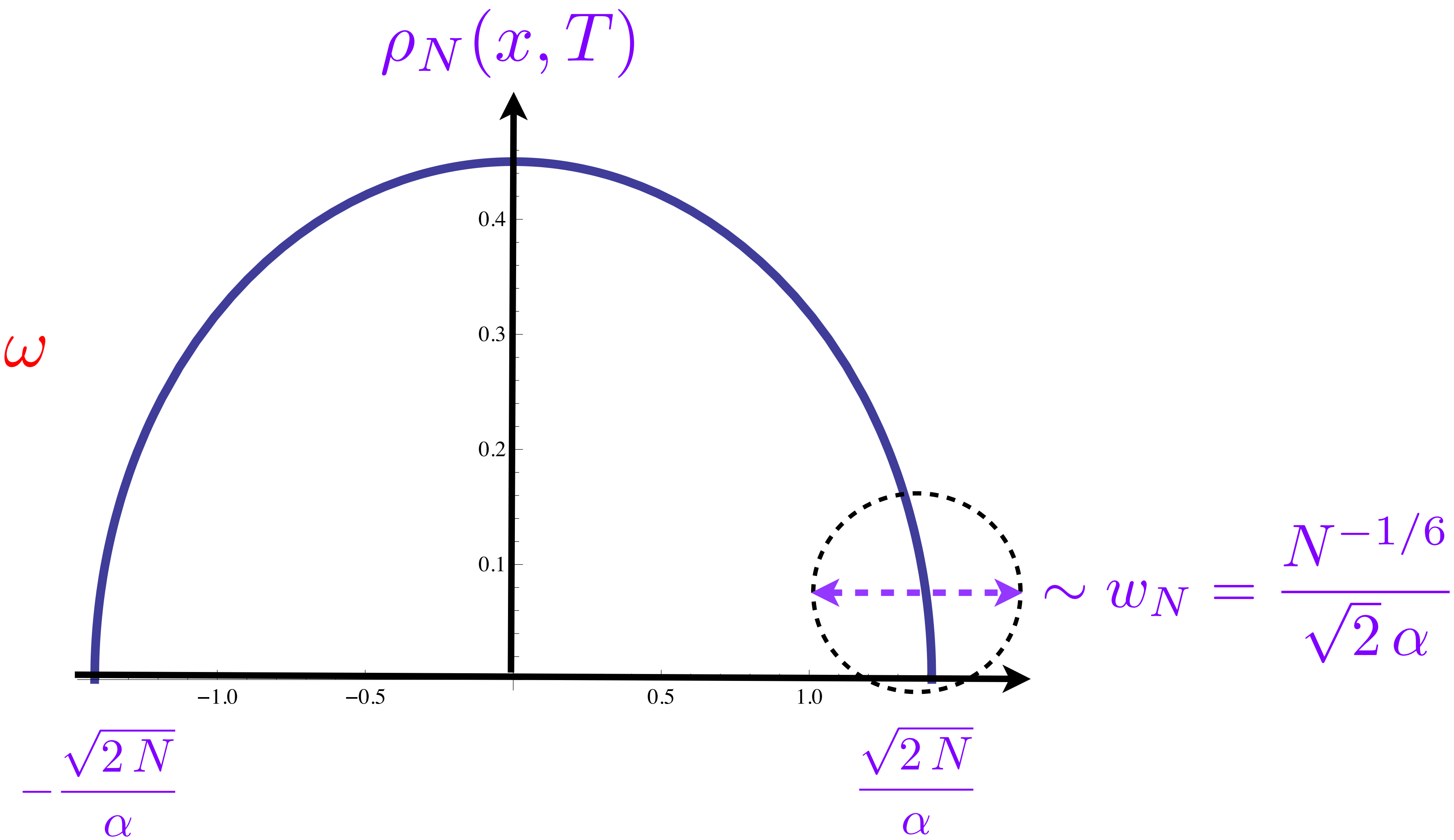

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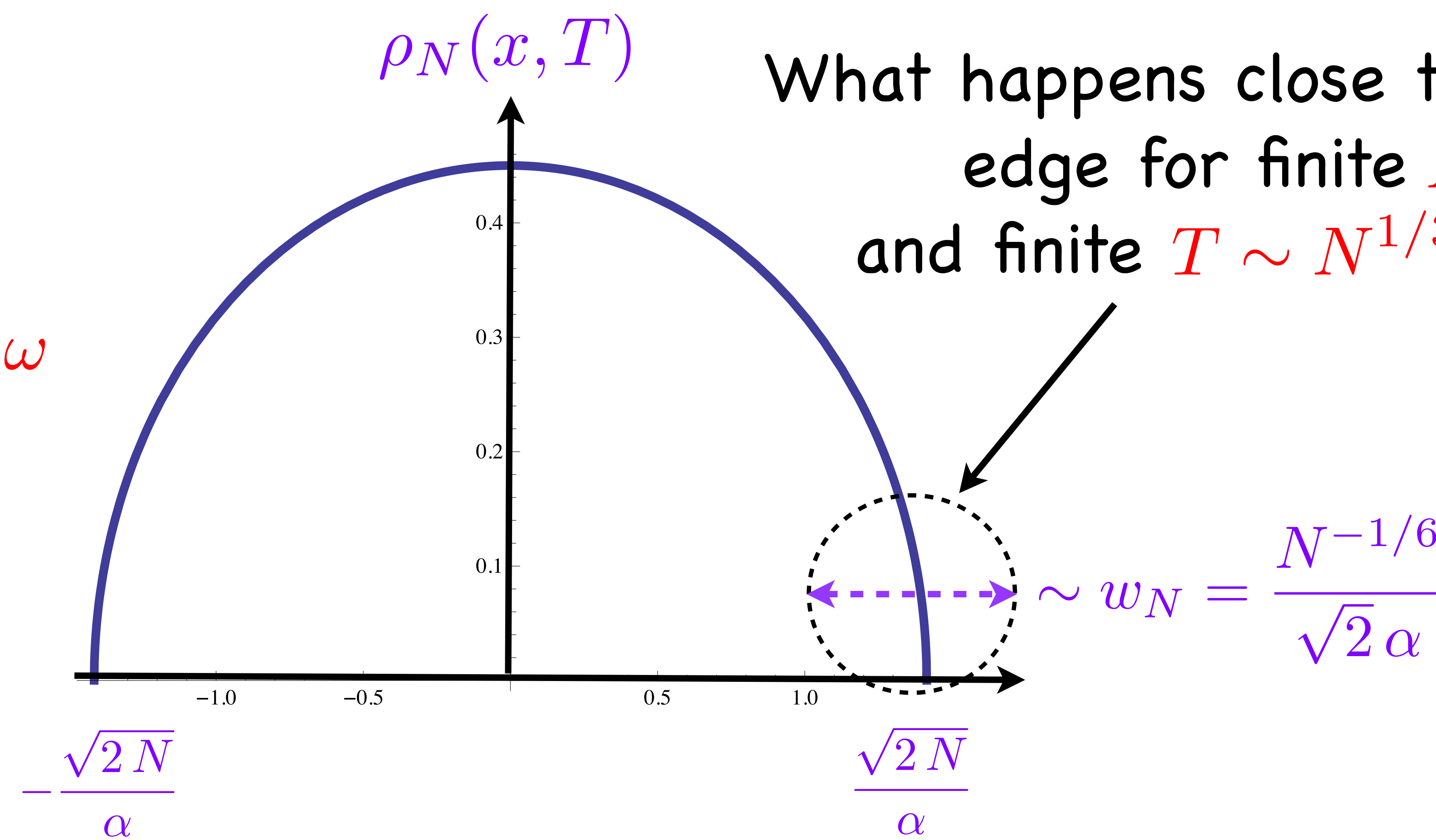


What happens close to the edge for finite $N$ and finite $T \sim N^{1/3}$ ?

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$\rho_N(x,T)$

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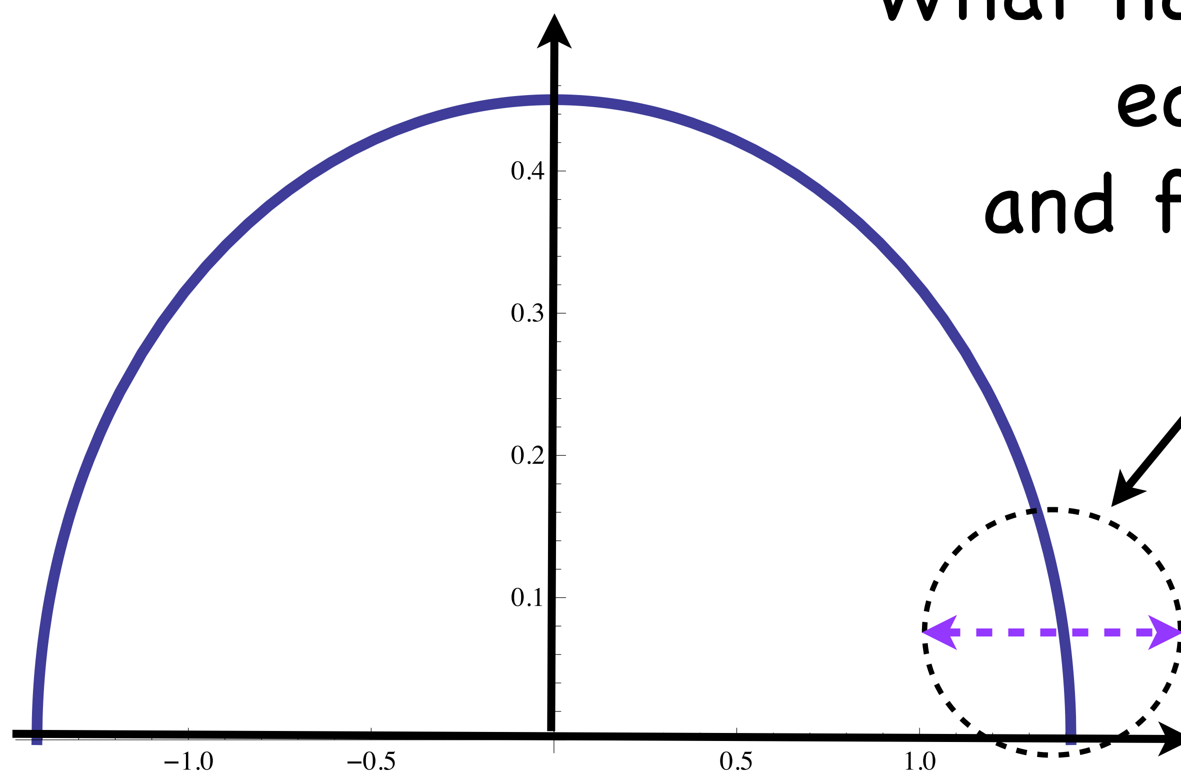
$T \sim b^{-1} N^{1/3} \hbar\omega$

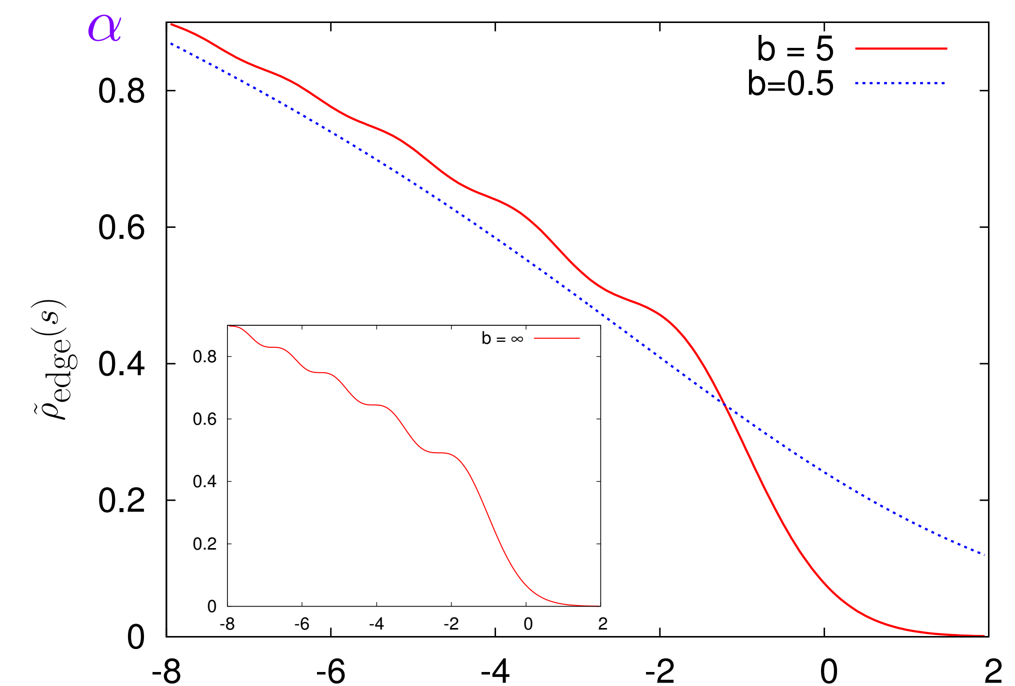



$\sim w_N = \dfrac{N^{-1/6}}{\sqrt{2}\,\alpha}$

$-\dfrac{\sqrt{2\,N}}{\alpha}$ $\qquad$ $\dfrac{\sqrt{2\,N}}{\alpha}$

$$\rho_N(x,T) \approx \frac{1}{N w_N} F_1\left(\frac{x-\sqrt{2N}/\alpha}{w_N}\right)$$

$$F_1(z) = \int_{-\infty}^{\infty} \frac{[Ai(z+u)]^2}{e^{-b\,u}+1}\,du$$



# Edge kernel for $N$ free fermions for T > 0

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Dean, Le Doussal, Majumdar, G. S. '15 generalization of the Airy-kernel

see also Johansson '07, Dong-Liechty '18

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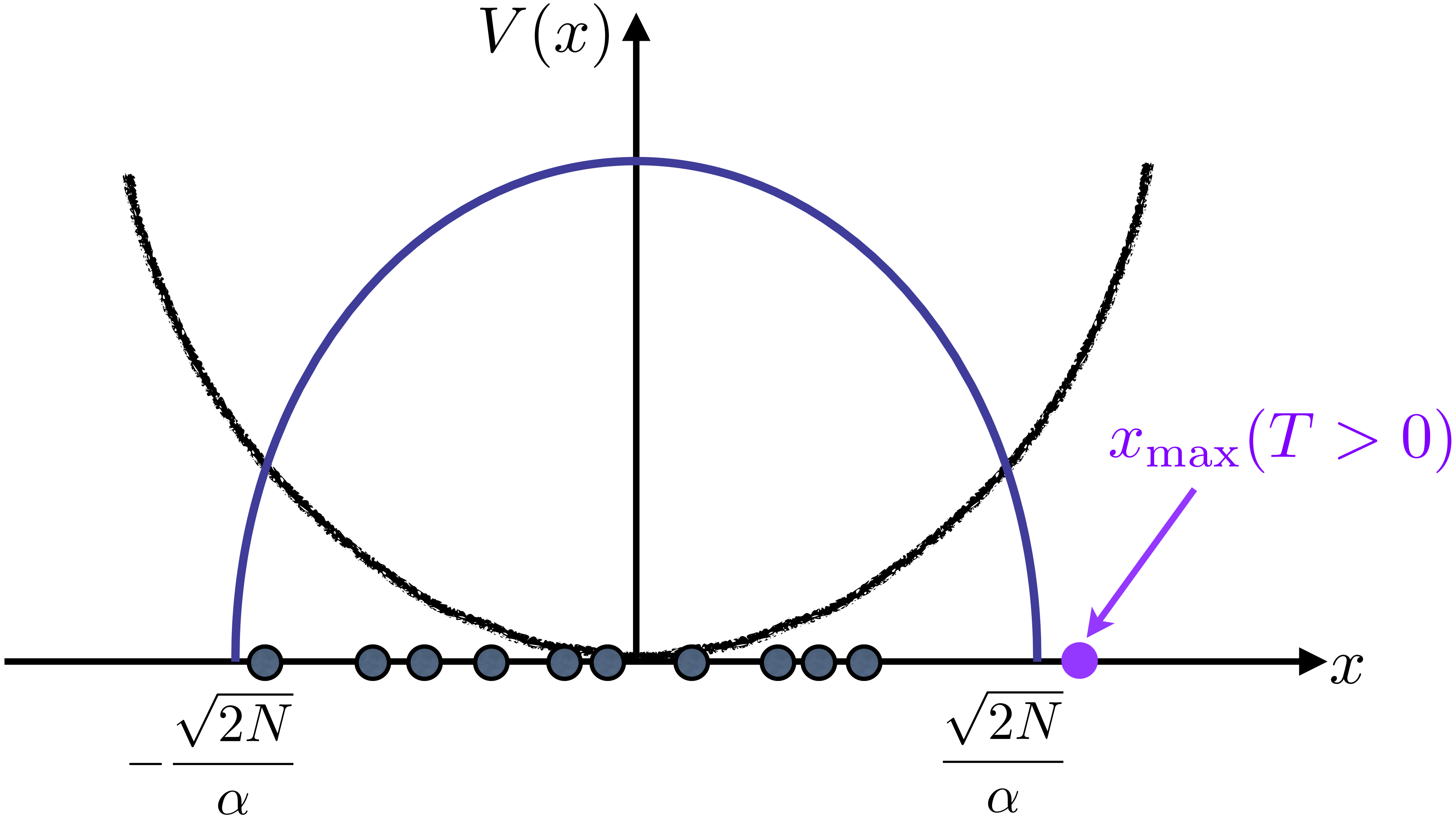
- Universal behavior, i.e., independent of the confining potential

$V(x) \sim |x|^p$ Dean, Le Doussal, Majumdar, G. S. '16

# Position of the rightmost fermion at finite but low $T$

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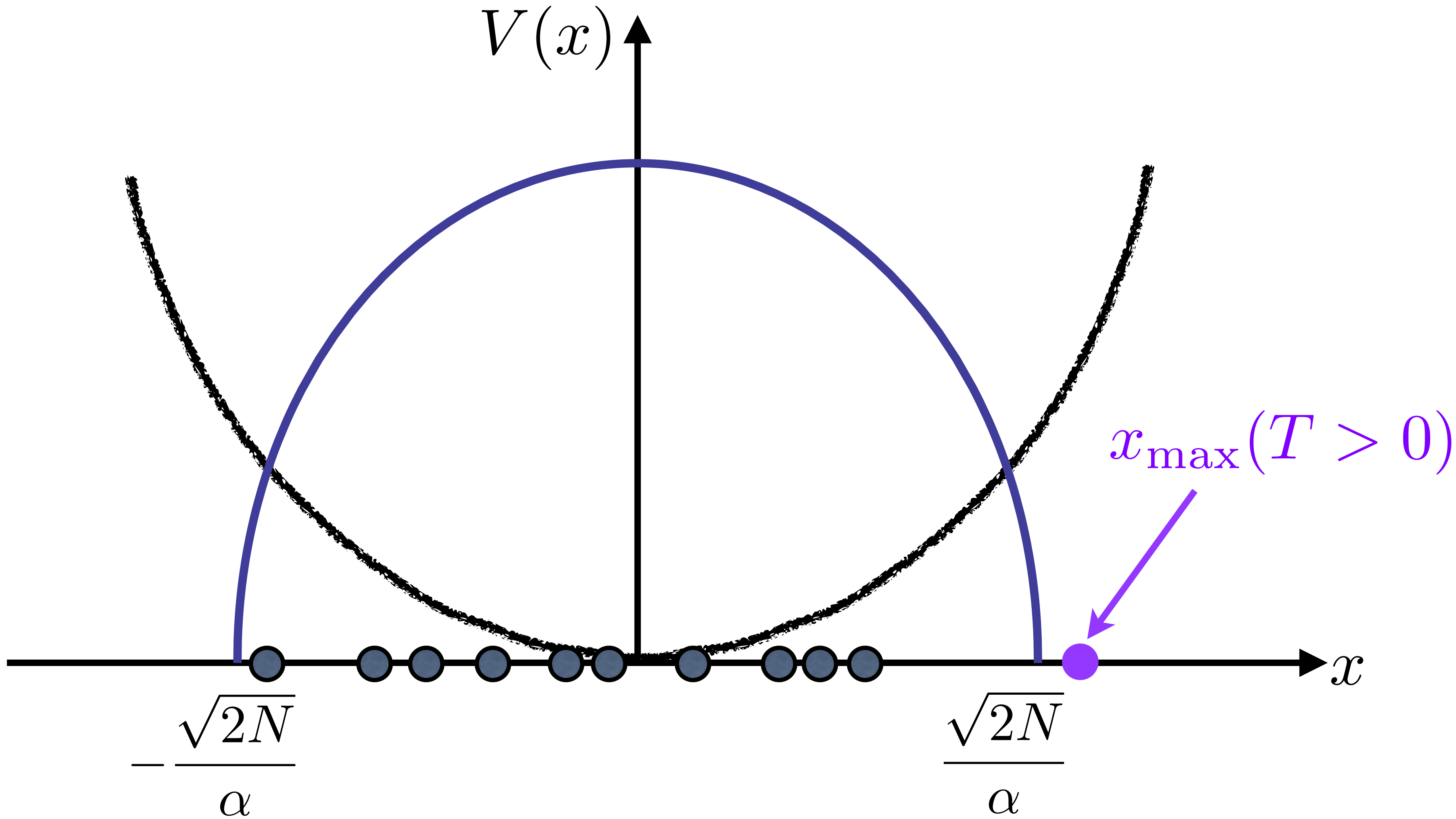


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- Fluctuations of $x_{\max}(T>0)$ — Dean, Le Doussal, Majumdar, G. S. '15

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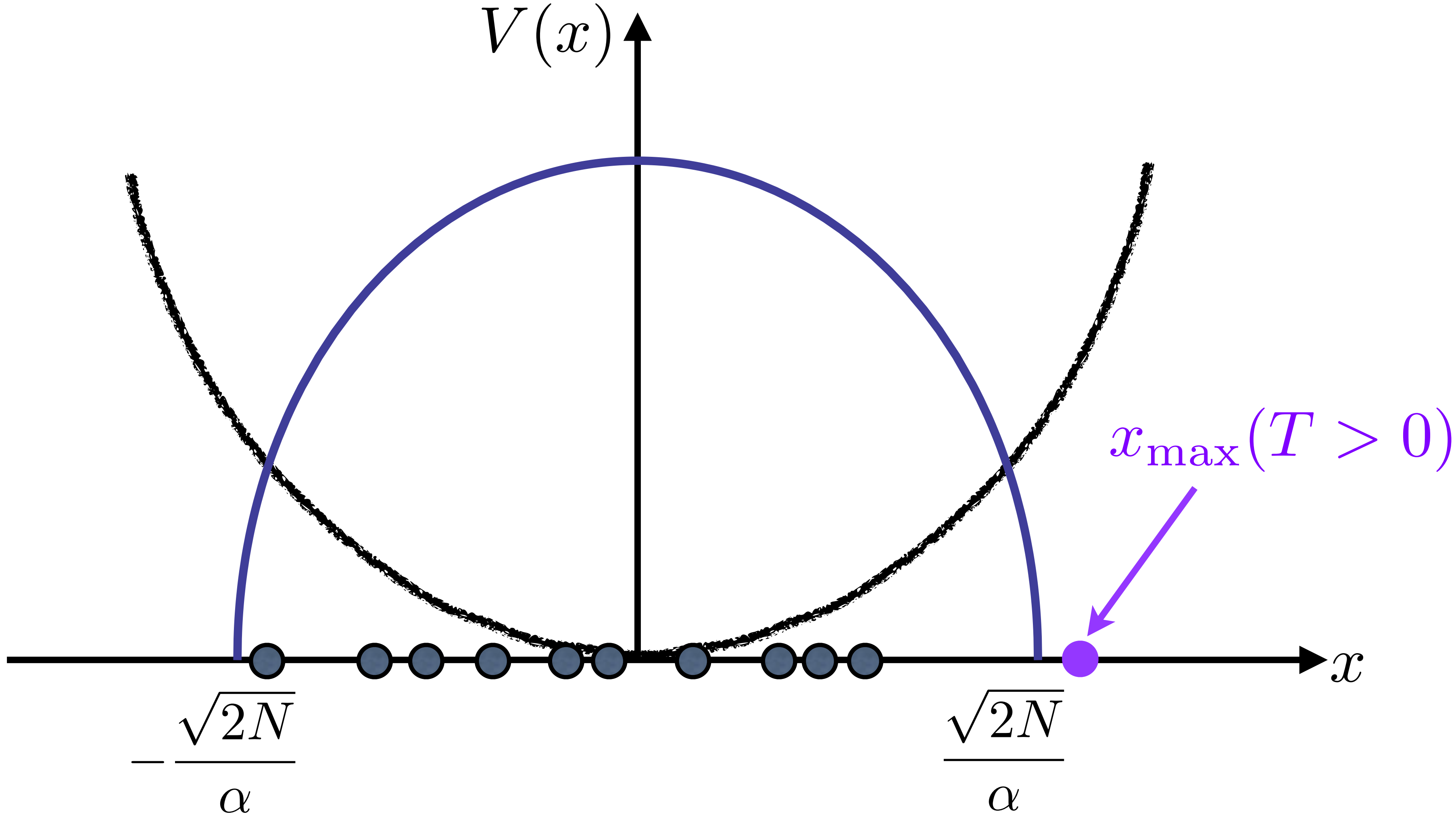


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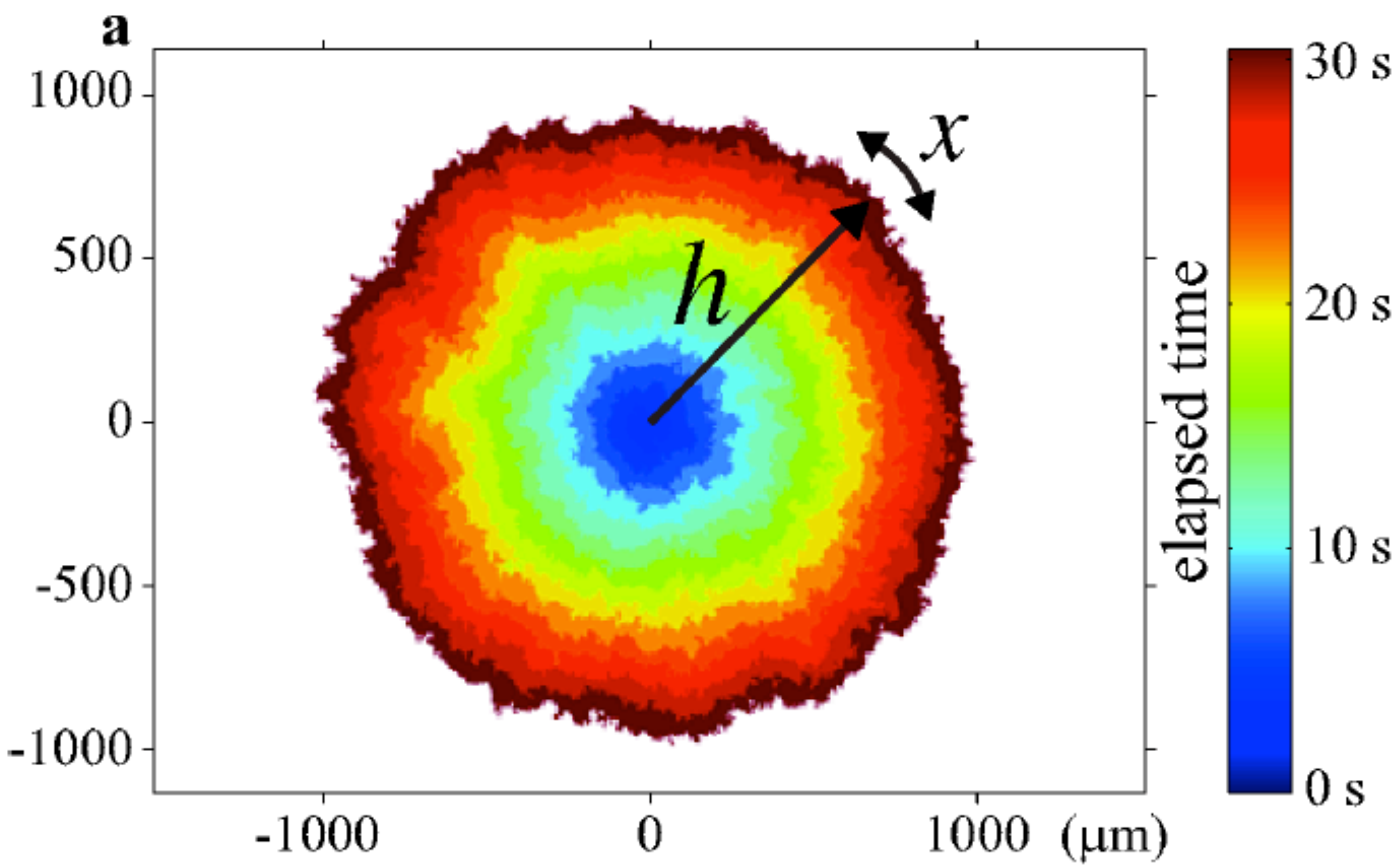
⟹ finite T generalization of the Tracy-Widom distribution

# Kardar-Parisi-Zhang (KPZ) equation at finite time

- KPZ equation in 1+1 dimensions in a curved geometry

(with dimensionless parameters) $\partial_t h = \partial_x^2 h + (\partial_x h)^2 + \eta(x,t)$




$\langle \eta(x,t)\eta(x',t')\rangle = \delta(x-x')\delta(t-t')$

from K. Takeuchi et al., Sci. Rep. '11

(experiments on liquid crystal)

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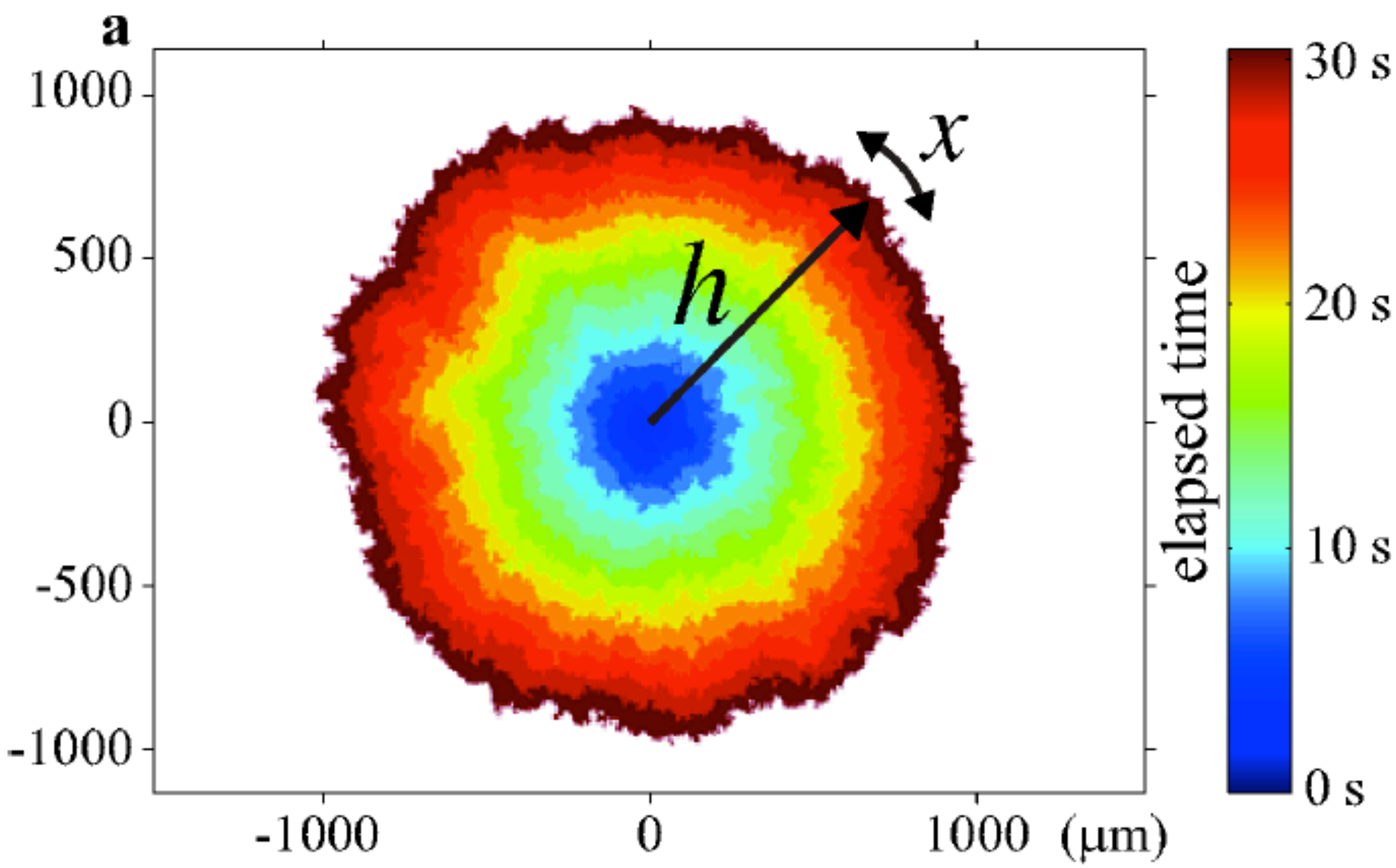


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- Exact solution of the KPZ equation in 1+1 dim. in a curved geometry

Sasamoto, Spohn '10/Calabrese, Le Doussal, Rosso '10/Dotsenko '10/ Amir, Corwin, Quastel '11 Imamura, Sasamoto, Spohn '13

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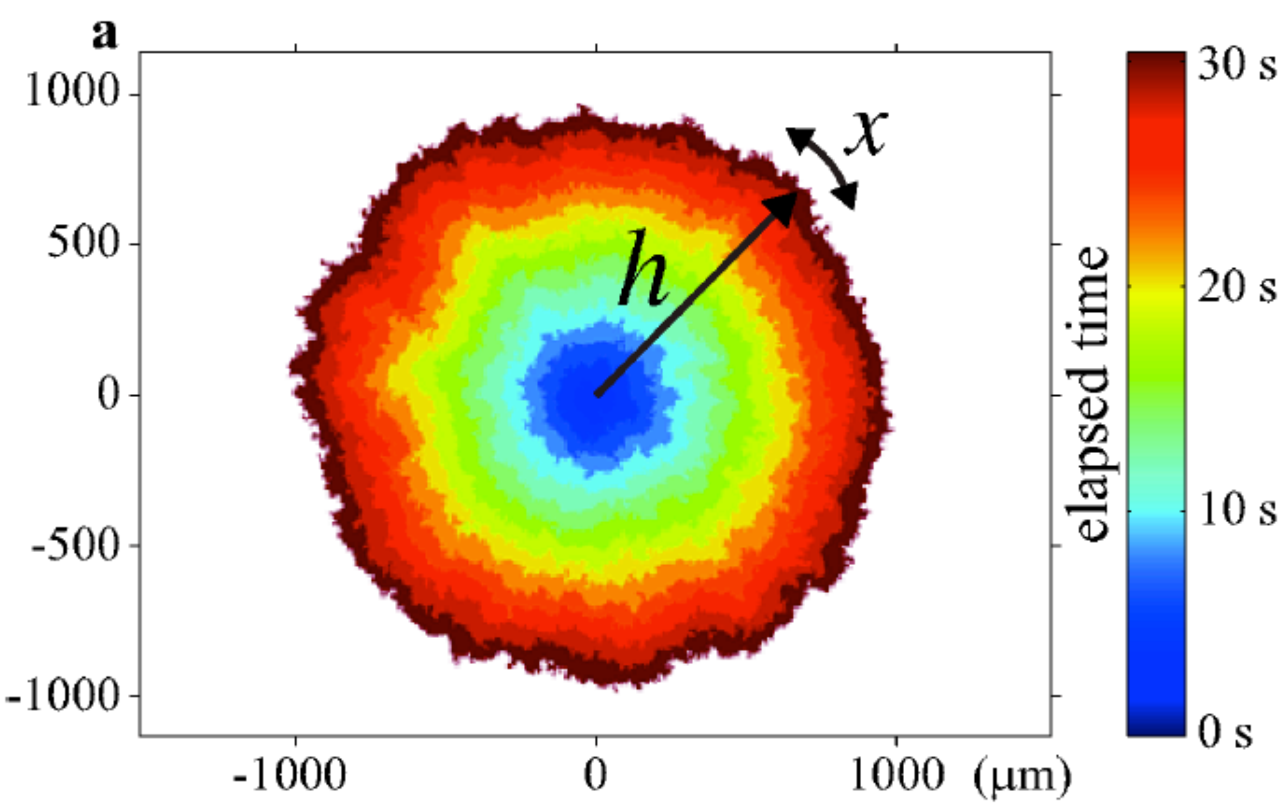


- Time-dependent generating function of the height field

$$g_t(s) = \langle \exp(-e^{h(0,t)+\frac{t}{12}-st^{1/3}})\rangle \ , \ g_t(s) = \det(I - P_s K_{\rm KPZ} P_s)$$

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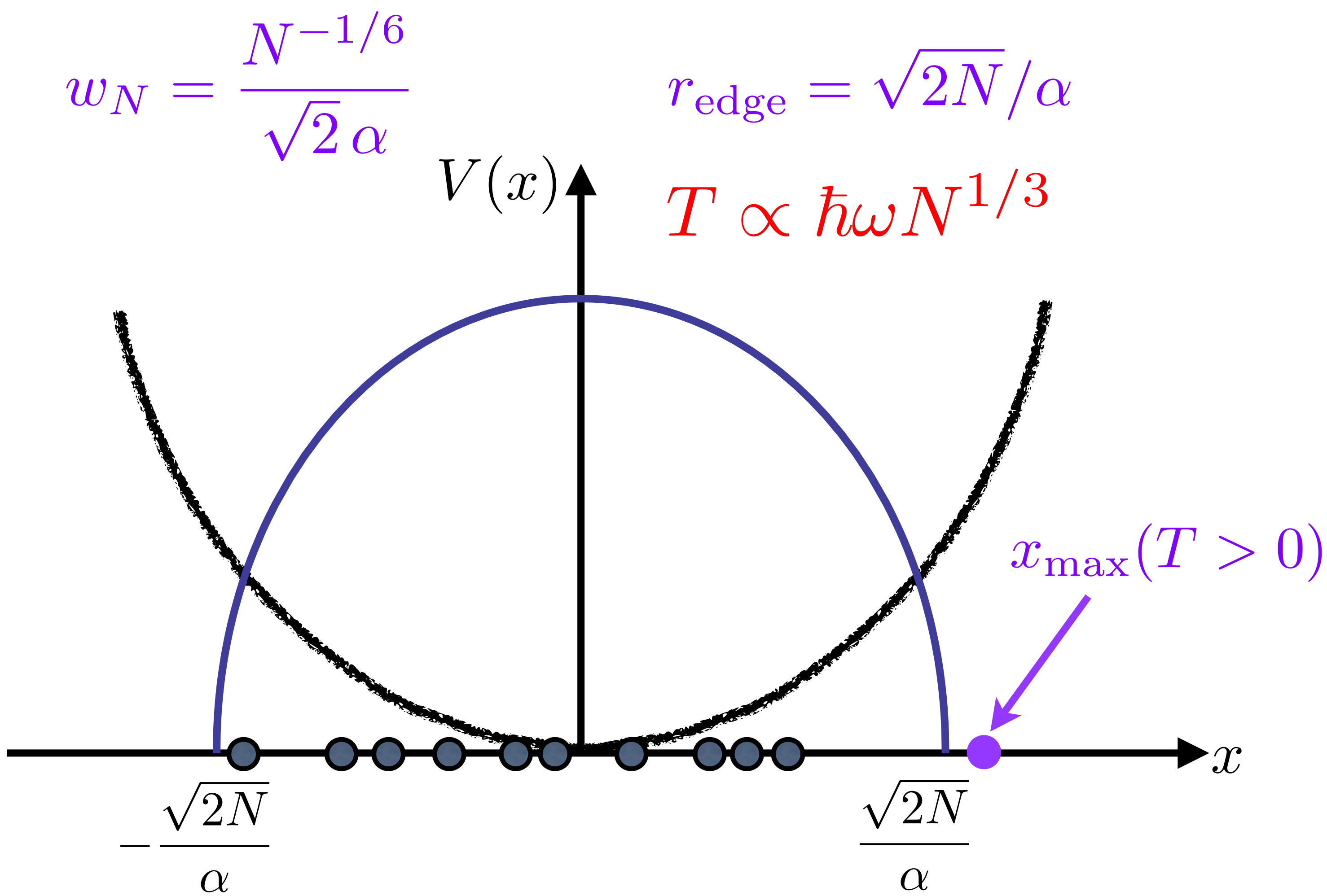
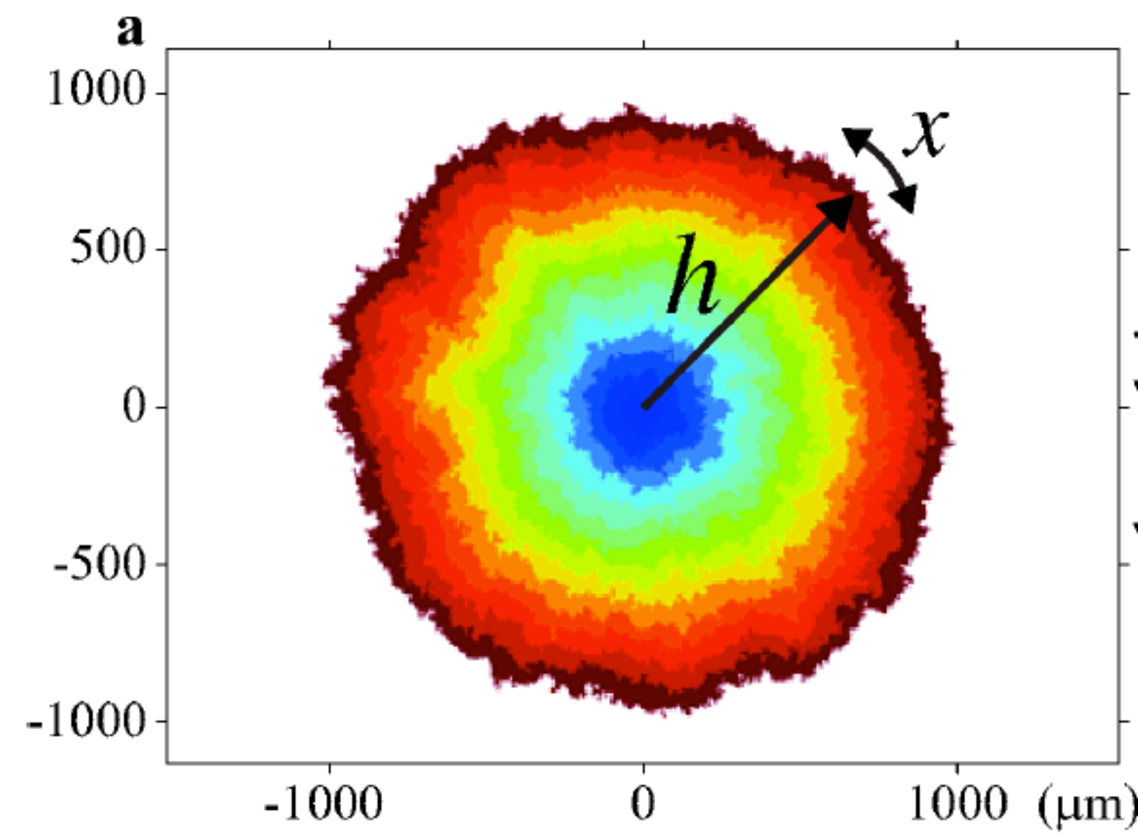
→ formal connection between the two problems with $1/T \Longleftrightarrow t^{1/3}$

Dean, Le Doussal, Majumdar, G. S. '15

# Connection between fermions at finite temperature and KPZ at finite time

Dean, Le Doussal, Majumdar, G. S., PRL '15



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Dean, Le Doussal, Majumdar, G. S., PRL '15

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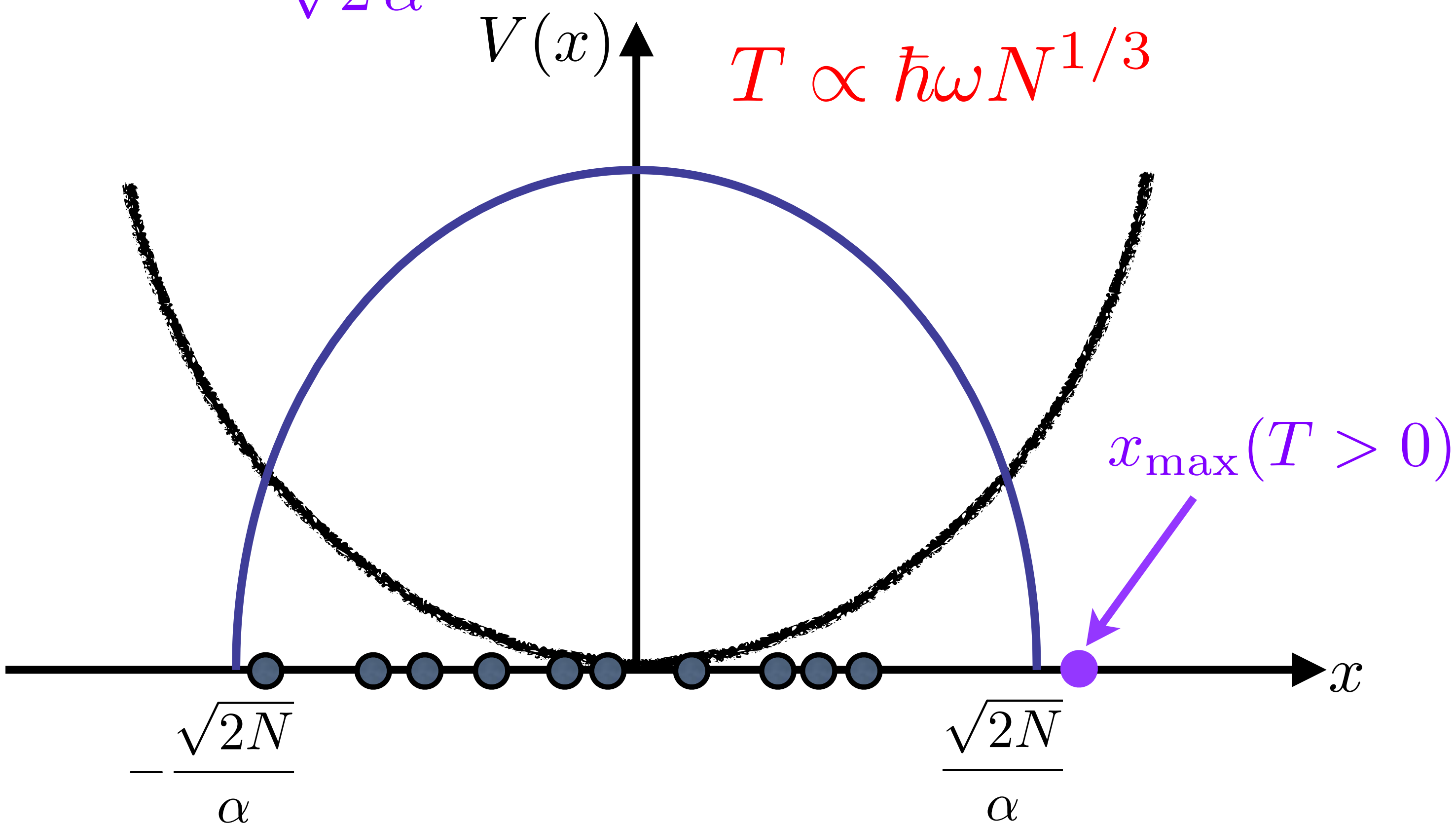
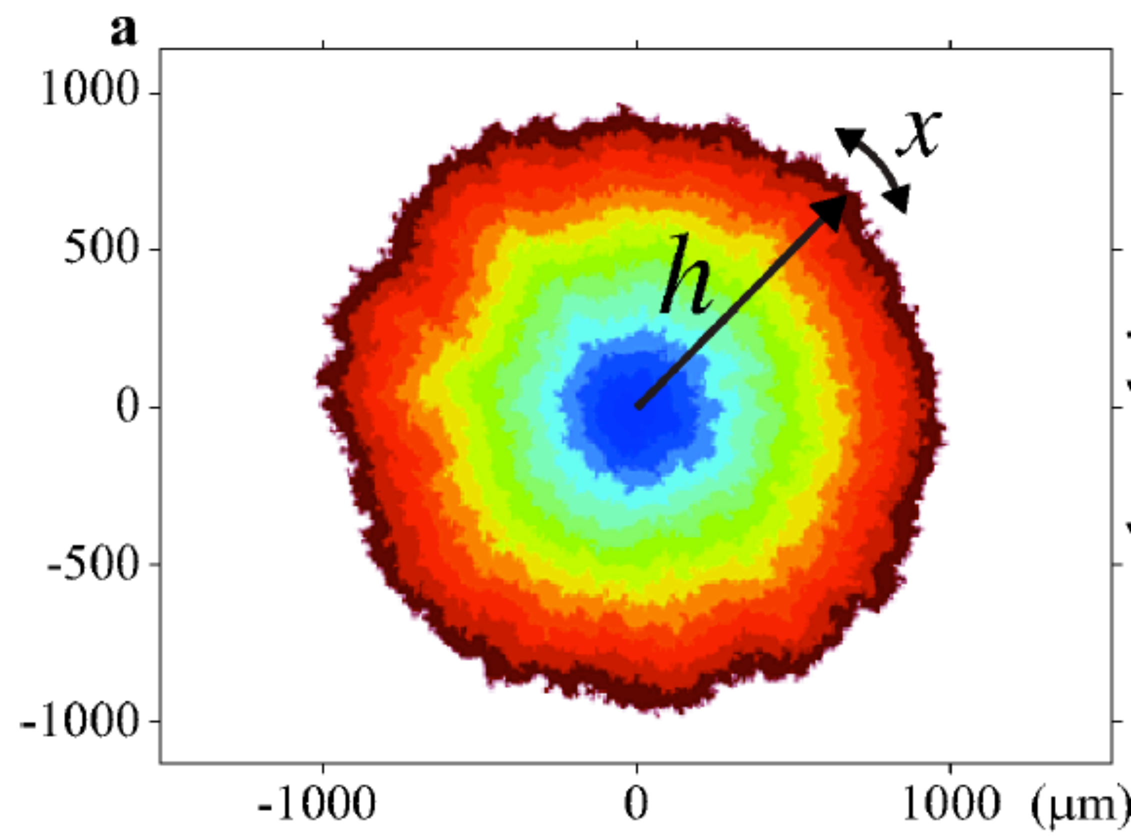
$$T \propto \hbar\omega N^{1/3}$$




$$\lim_{N\to\infty} \frac{x_{\max}(T) - r_{\rm edge}}{w_N} \overset{\text{in law}}{=} \frac{h(0,t) + v_\infty t + G}{t^{1/3}}$$

$$1/T \Longleftrightarrow t^{1/3}$$

Gumbel variable independent of $h(0,t)$

What happens in $d > 1$?

# Free fermions in a d-dimensional harmonic trap (T=0)

- Single particle Hamiltonian

$$\hat{H} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_d^2}\right) + \frac{1}{2}m\omega^2\,\underbrace{\left(x_1^2 + \cdots x_d^2\right)}_{r^2}$$

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- Global density (at T=0)

$$\rho_N(\mathbf{x}) \approx \frac{1}{N}\left(\frac{m}{2\pi\hbar^2}\right)^{d/2}\frac{[\mu - \frac{1}{2}m\omega^2 r^2]^{d/2}}{\Gamma(d/2+1)}$$

with $\mu \approx \hbar\omega[\Gamma(d+1)\,N]^{1/d}$

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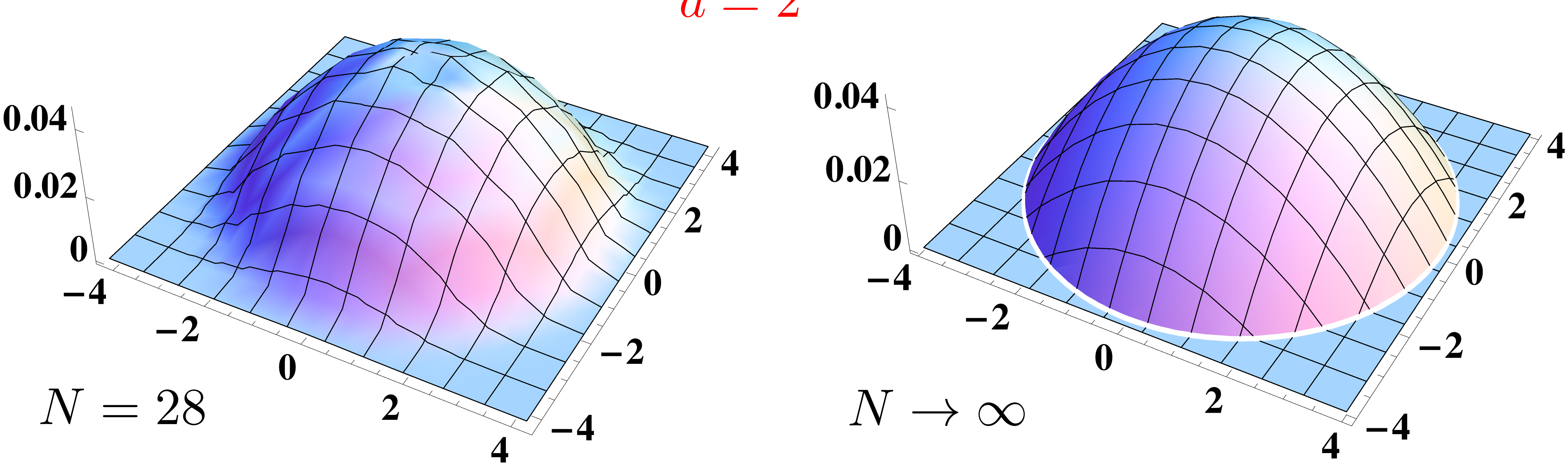
- Global density (at T=0)

$$\rho_N(\mathbf{x}) \approx \frac{1}{N}\left(\frac{m}{2\pi\hbar^2}\right)^{d/2} \frac{[\mu - \frac{1}{2}m\omega^2 r^2]^{d/2}}{\Gamma(d/2+1)}$$

with $\mu \approx \hbar\omega[\Gamma(d+1)\, N]^{1/d}$

$d = 2$



# Free fermions in a d-dimensional harmonic trap (T=0)

- Edge density of free fermions

$$\rho_{\rm edge}(\mathbf{x}) \approx \frac{1}{N}\frac{1}{w_N^d} F_d\left(\frac{r - r_{\rm edge}}{w_N}\right)$$

with $w_N = b_d\, N^{-\frac{1}{6d}}$ and $F_d(z) = \frac{1}{\Gamma(\frac{d}{2}+1)2^{\frac{4d}{3}}\pi^{\frac{d}{2}}}\int_0^\infty du\; u^{\frac{d}{2}} Ai(u + 2^{2/3}\, z)$

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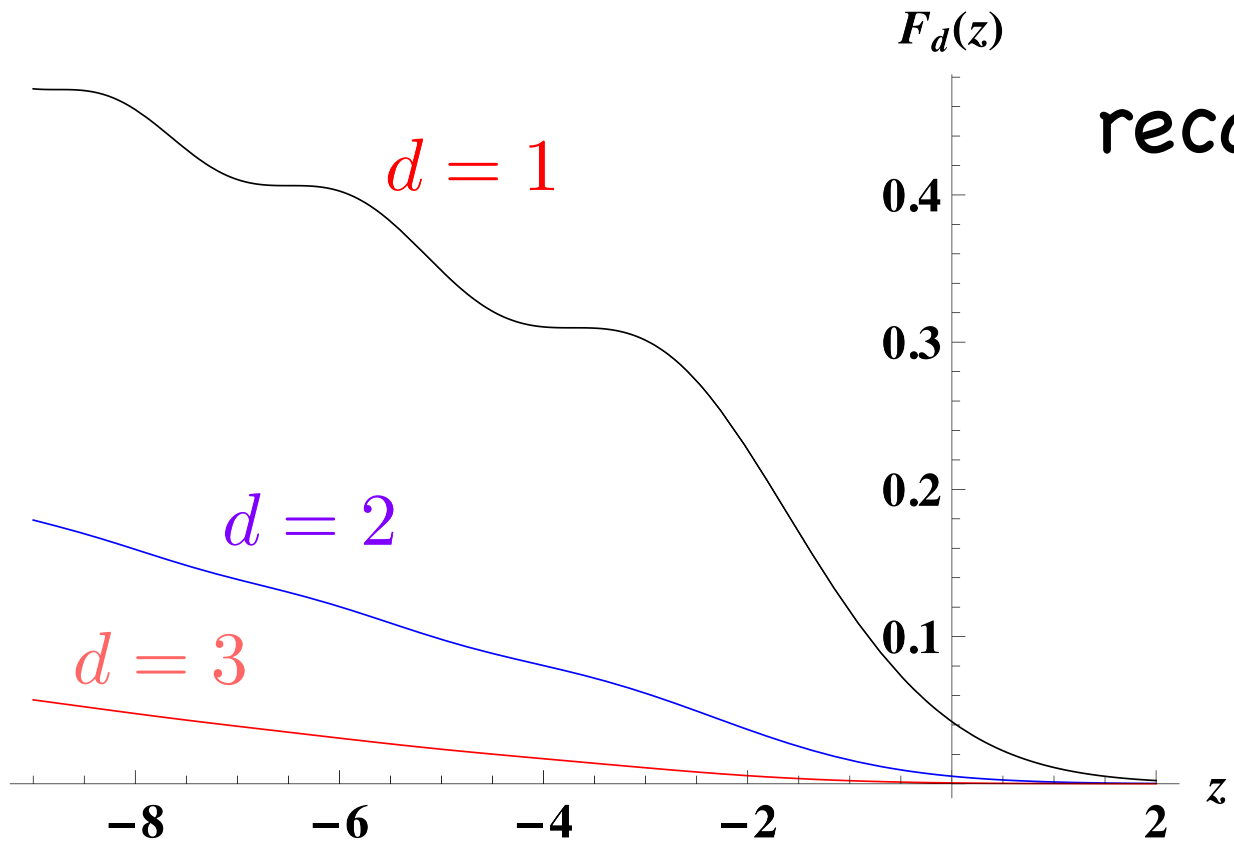


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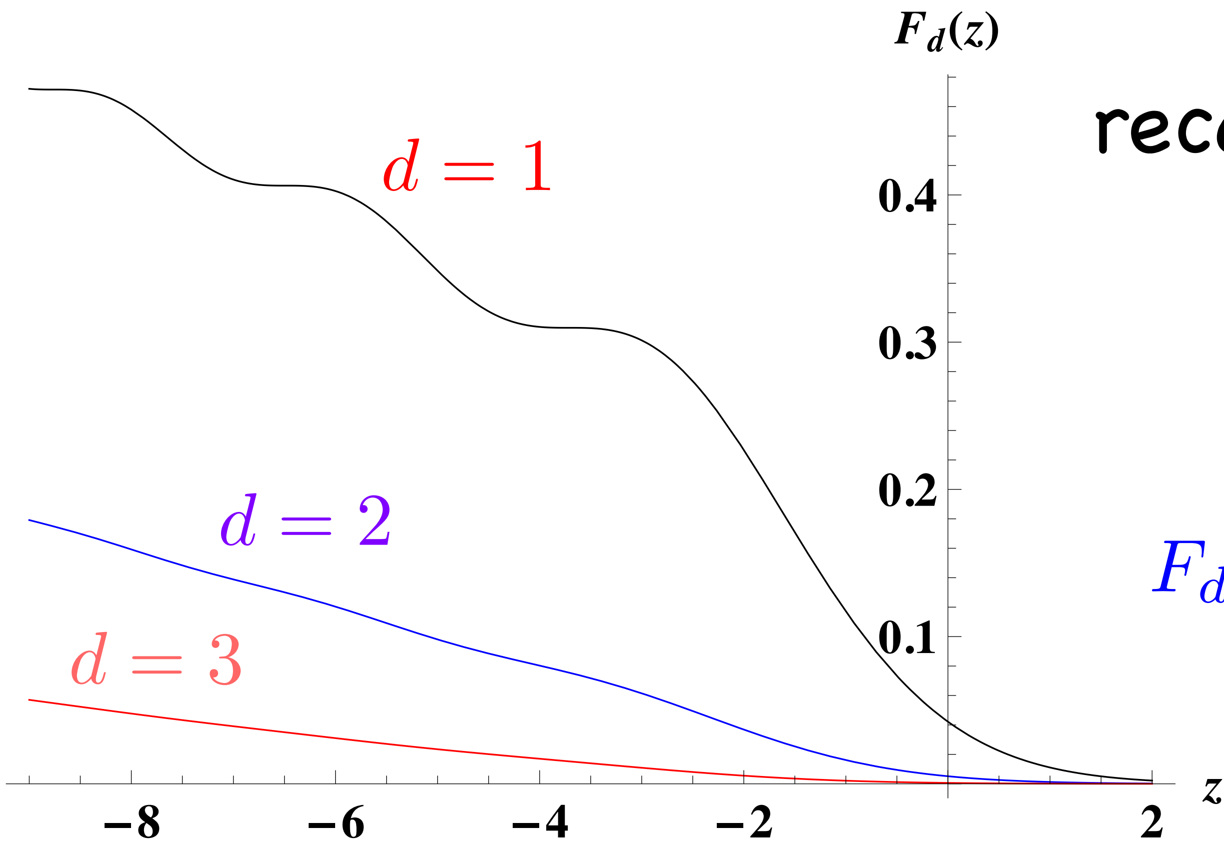


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$$F_d(z) \approx \begin{cases} (8\pi)^{-\frac{d+1}{2}}\, z^{-\frac{d+3}{4}}\, e^{-\frac{4}{3}\, z^{3/2}} \text{ as } z \to \infty \\ \frac{(4\pi)^{-\frac{d}{2}}}{\Gamma(d/2+1)}\, |z|^{\frac{d}{2}} \quad \text{as} \quad z \to -\infty \end{cases}$$

# Free fermions in a d-dimensional harmonic trap (T=0): limiting correlation kernels

$$K_N(\mathbf{x},\mathbf{y}) = \sum_{\mathbf{k}} \theta(E_F - \epsilon_{\mathbf{k}})\psi_{\mathbf{k}}(\mathbf{x})\psi_{\mathbf{k}}(\mathbf{y})$$

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with

$$\mathcal{K}_{\rm edge}(\mathbf{a},\mathbf{b}) = \int \frac{d^d q}{(2\pi)^d} e^{-i \mathbf{q}\cdot(\mathbf{a}-\mathbf{b})} Ai_1\left(2^{\frac{2}{3}} q^2 + \frac{a_n + b_n}{2^{1/3}}\right)$$

$a_n = \mathbf{a} \cdot \mathbf{r}_{\rm edge}/r_{\rm edge}$ and $b_n = \mathbf{b} \cdot \mathbf{r}_{\rm edge}/r_{\rm edge}$

$$Ai_1(z) = \int_z^\infty Ai(u) du$$

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$$\hat{H}(\hat{\mathbf{p}}, \hat{\mathbf{r}}) = \frac{\hat{\mathbf{p}}^2}{2} + \frac{\hat{\mathbf{r}}^2}{2} - \Omega \hat{L}_z$$

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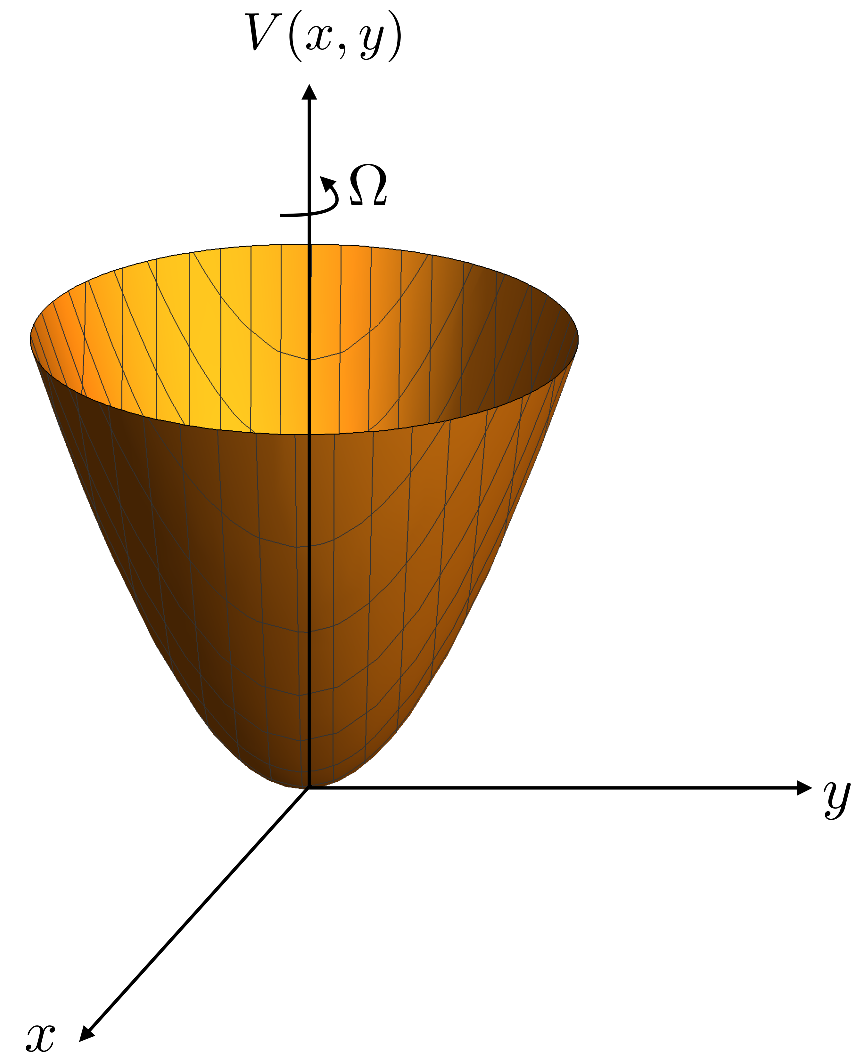

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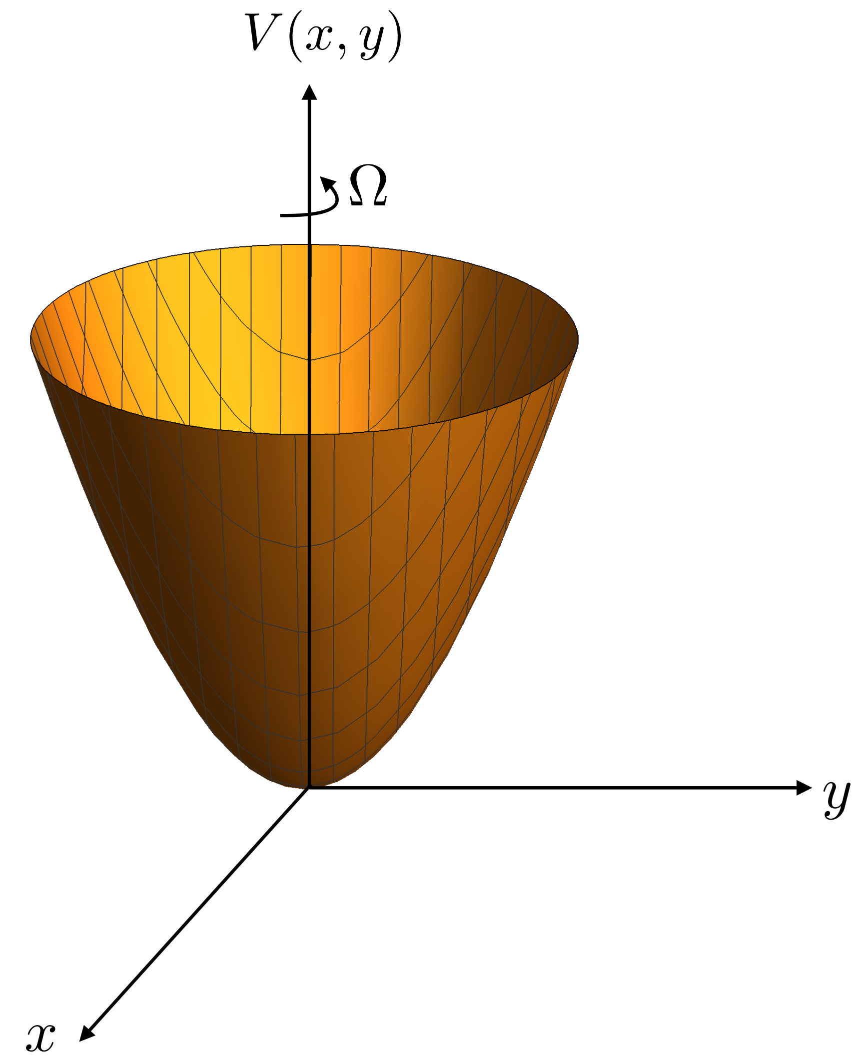


for $1 - \frac{2}{N} < \Omega < 1$

- Joint PDF of the positions $z_i$ of N non-interacting fermions at T=0

Lacroix-A-Chez-Toine, Majumdar, G. S., PRA '19

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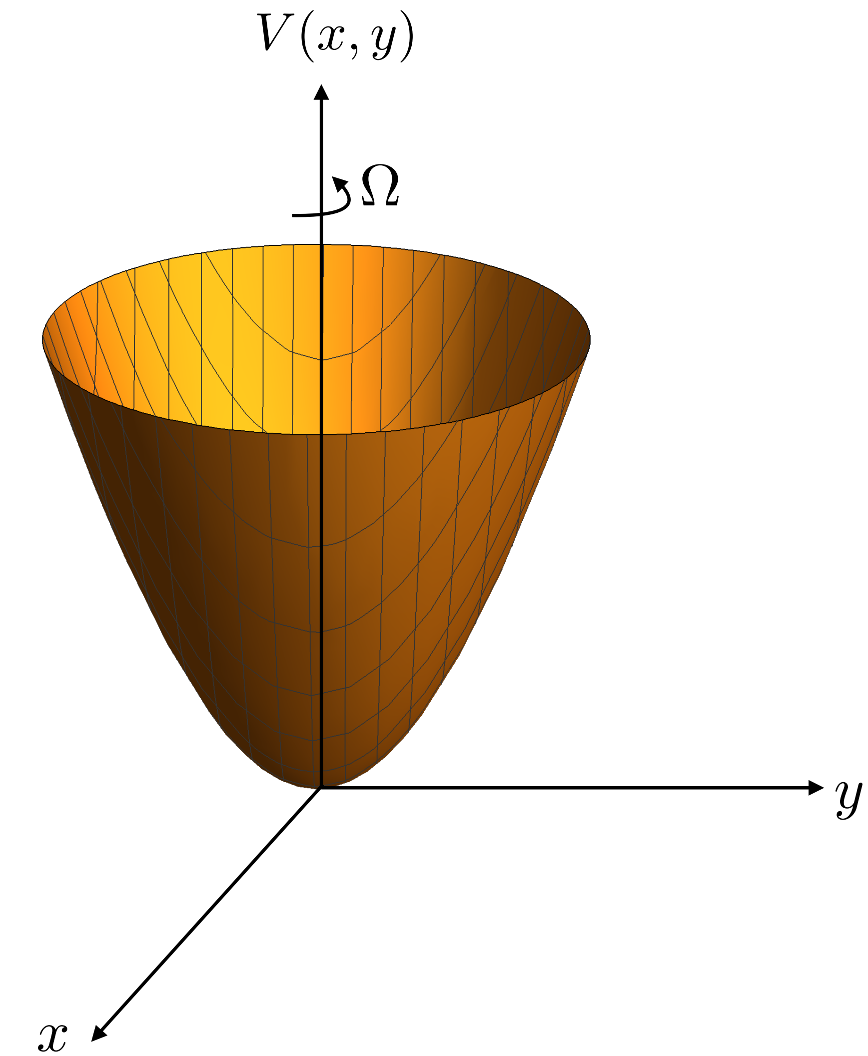


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What about the interactions ?

# Calogero-Sutherland-Moser model and random matrices

- Interacting fermions in d=1

$$\hat{\mathcal{H}}_N = \sum_{i=1}^{N} \left[ \frac{\hat{p}_i^2}{2} + \frac{\hat{r}_i^2}{2} \right] + \sum_{i<j} V(\hat{r}_i - \hat{r}_j)$$

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⇒ Gaussian $\beta$-ensemble of random matrices

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- Can one observe these properties in cold atoms experiments ?

# Some open questions

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- Effects of interactions ?

- Effects of disorder/impurities ?

- Dynamics of non-interacting fermions (« quantum quench »)